DISTRIBUTIVITY IN LATTICES

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1. Introduction. A lattice L is infinitely (join) distributive if $a \cap \cup_B b = \cup_B(a \cap b)$ whenever the indicated joins exist in L. Clearly infinite distributivity implies ordinary distributivity. On the other hand it is easy to give examples of distributive lattices which are not infinitely distributive. For example, the rational integers under the relation of division form a distributive lattice which is not infinitely distributive. However for complemented lattices, as observed by Tarski [8] and von Neumann [6], distributivity implies infinite distributivity.

Another curious consequence of complementation is the following theorem due to Stone [7] and Glivenko [5]. Let a subset A of a Boolean algebra $\mathfrak{B}$ be called *normally closed* if A contains all lower bounds of its set of upper bounds. Then the collection of closed subsets of $\mathfrak{B}$ is a Boolean algebra under set inclusion and hence is infinitely distributive by the theorem mentioned above. On the other hand, Funayama [4] and Cotlar [2] have given examples of distributive lattices whose closed subsets form non-distributive (indeed, non-modular) lattices. In §6 we give an example of a lattice which is even infinitely distributive and whose lattice of normally closed subsets is non-modular.

The purpose of this paper is to formulate and study the principles upon which these results rest.

We first introduce a general class of distributive laws. Let ϕ be an imbedding operator (Ward [9]) on a lattice L. Thus ϕ is a closure operation (Birkhoff [1]) on L such that $\phi(a) = (a)$ where (a) denotes the set of all $x \leq a$. L is said to be *ϕ-distributive* if

$$a \cap \phi(S) = \phi(a \cap S)$$

for all $a \in L$ and all subsets S of L.

Thus if ϕ is the ideal operator, ϕ-distributivity is ordinary distributivity. Similarly, if ϕ is the complete ideal operator, then ϕ-distributivity is infinite distributivity. (A subset A of L is a *complete ideal* if $S \subseteq A$ and $\cup S$ exists in L, then $\cup S \in A$.)

The imbedding operators on L have a natural partial ordering defined by $\phi \geq \psi$ if and only if $\phi(S) \supseteq \psi(S)$ for all subsets S of L. The null operator under this partial ordering is the operator ω defined by

$$\omega(S) = \{x \mid x \leq s \text{ some } s \in S\}.$$

It is easily shown that every lattice L is ω-distributive. On the other hand, the unit operator ν with respect to the partial ordering is the normal operator

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