THE INVERSION OF SOLUTIONS OF THE HEAT EQUATION FOR THE INFINITE ROD

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Consider the heat equation

$$u_{xx} = u_t \tag{1}$$

in the region R: $(0 < t < T, -\infty < x < \infty)$. $u(x,t)$ is a solution of (1) if u_{xx} and u_t exist at every point of R and satisfy (1). It is a result of the classical theory of the heat equation [10] that if $\phi(x)e^{-cx^2} \varepsilon L$ for some c, then the integral

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-r)^2/4t} \phi(r)\, dr \tag{2}$$

exists for $0 < t < 1/4c$, is a solution of (1), and $\lim_{t \to 0+} u(x,t) = \phi(x)$ a.e..

We shall treat the following problems:

(A) Suppose we have a solution of (1) in $(0 < t < T, -\infty < x < \infty)$ represented by (2). Consider $\lim_{t \to T-} u(x,t)$. This may or may not exist. We shall be interested in whether this limit can exist for all x even though (2) diverges for $t = T$. A solution of the form (2) will be exhibited for which (2) converges for $0 < t < T$, diverges for $t \geq T$, but such that the solution in $0 < t < T$ can be extended continuously to the region $t \geq T$.

(B) It is known that if (2) converges in $0 < t < T$, (2) represents an entire function in x. If (2) diverges for $t = T$ but $\lim_{t \to T-} u(x,t)$ exists, the question arises of whether this limit is also entire. It will be shown that this is the case under general conditions.

(C) If in (2), $\lim_{t \to T-} u(x,t) = g(x)$, then it is natural to ask whether, knowing $g(x)$, $u(x,t)$ can be determined. This leads to the problem of solving the backward heat equation $v_{xx} + v_t = 0$ in the region R under the condition $\lim_{t \to 0+} v(x,t) = g(x)$. This will be solved when $g(x)$ satisfies certain order conditions.

(D) The methods used in answering (C) will lead to a relationship between summation of Fourier integrals and the uniqueness theorems of the heat equation. This will lead to a theorem on the summation of Fourier integrals.

1. **The analytic nature of the solutions.** S. Bernstein [2] has a general treatment of the analytic character of certain second order partial equations and M. Gevrey [7] has a treatment of certain parabolic equations. The results below have the advantage of being applicable to the limit problem described above and of requiring somewhat weaker hypotheses.

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