THE GEOMETRY OF NORMED LATTICES

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1. Introduction. If it is possible to associate with each element x of a lattice L a real number $| x |$ such that $| x | > | y |$ for $x \supset y$, $x \neq y$ and $| x | + | y | = | x \cup y | + | x \cap y |$, then such an association defines a modular functional over the lattice. Such an association is only possible in the event that the lattice is modular. The number $| x |$ is called the norm of x and the lattice is referred to as a normed lattice or alternatively as a metric lattice. It is then possible to introduce into the lattice a "distance" by assigning to each pair of elements x and y of L the real nonnegative number $d(x,y) = | x \cup y | - | x \cap y |$. It is readily verified that the resulting distance space is metric. The structure is, then, at once a lattice and a metric space, the two being connected through the medium of the norm. We reserve the symbol $D(L)$ for the metric space associated with the normed lattice L. Where not explicitly stated otherwise, L will stand for a normed lattice. It is clear that imposing algebraic restrictions on L will, in general, affect in some way the geometry of $D(L)$ and conversely. Our concern is with this interplay.

Glivenko [4], [5] initiated this study in 1936 and several papers, notably [3] and [7] have dealt with it since then. (An account of this aspect of lattice theory is contained in a forthcoming book by L. M. Blumenthal.) One of the principal problems in this study is to decide whether a given metric space is the associated metric space of some normed lattice. For example, no normed lattice can have the Euclidean plane as its associated metric space [3].

Glivenko [4] and Smiley and Transue [7] have formulated conditions which enable one to decide whether a given metric space is the associated metric space for a lattice with *nonnegative norm and with a first element.* Glivenko's condition is that the space be "almost ordered" (See Definition 2.6). Smiley and Transue characterize the space in terms of transitivities on the metric betweenness together with the existence of certain pseudolinear quadruples.

In the first part of this paper we extend the Smiley-Transue and Glivenko results to any normed lattices. (This solves problem 66, page 139 of Birkhoff [1]). We lean heavily, however, on their approach. In the second part we consider the general problem of reconstructing the lattice L from a knowledge of its geometric "skeleton" $D(L)$. In particular we consider some of the geometry of ideals and various special normed lattices.

2. Preliminaries. Definition 2.1. In a metric space, the point y is *between* the points x and z (written xyz) provided $d(x,y) + d(y,z) = d(x,z)$. The three points are *linear* if one is between the other two. It is convenient to relax the usual demand that y be distinct from x and z.

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