SUMMABILITY: THE INSET, REPLACEABLE MATRICES, THE BASIS IN SUMMABILITY SPACE

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1. **Preliminaries.** The purpose of this paper is to introduce the inset, a subspace of the field of a summability matrix, and to exhibit some of its properties. The device of the inset provides a tool for obtaining information concerning the "size" of the convergence fields of matrices. See, for example, Theorems 9, 10, and the second corollary to Theorem 2. It yields results on the following problem: does there exist a regular matrix which sums the same sequences as a given matrix? (If so, we say that the matrix is *replaceable.*) See [8], also Theorem 3, and Lemmas 8 to 11. In Theorem 4 we show an intimate connection between the inset and the property of certain matrices that the usual basis for the space of convergent sequences is a basis for the summability field. The corollaries to Theorems 2 and 4 give new properties of Cesàro, Hölder, and Riesz typical means matrices. An application to the existence of functionals on a normed space is given in Theorem 13, a remark on consistency in Theorem 12, and a remark on the continuity of a certain function in Theorem 15. Theorem 14 provides a partial answer to the following question: what are the conditions on a sequence $\{t_n\}$ in order that the series $\sum t_n x_n$ shall be convergent whenever x is a sequence which is summable by a given method? There is an application to a system of linear equations in infinitely many unknowns in Theorem 16.

In some places, the essential point is an inversion in the order of summations. This is justified in a non-classical way in Lemma 13.

In §2 we give notation and definitions; in §3 will be found listed Theorems 1 to 12; 15 lemmas, some of independent interest, are given in §4; proofs for the theorems are given in §5; four more theorems are given in §6.

2. **Notation and definitions.** Let $A = (a_{nk})$ be a matrix of complex numbers $(n, k = 1, 2, \cdots)$, and let (A) be the set of sequences summable A, that is, the set of those sequences x for which $A(x) = \lim A_n(x)$ exists, where $A_n(x) = \sum_k a_{nk} x_k$; (A) is then called the summability field of A, or, simply, the field of A. We shall always understand that (A) is a normed space with the norm defined by $\| x \| = \sup_n | A_n(x) |$. A matrix A is *normal* if, for each index n, $a_{nn} \neq 0$ and $a_{nk} = 0$ when $k > n$. If A is normal, (A) is a Banach space. Otherwise, with a different topology, (A) is an FK space in the sense of Zeller [10] . A is called *stronger* than B if $(A) \supseteq (B)$; two matrices are called *comparable* if one is stronger than the other.

For each k, let a_k denote the column limit, $\lim_n a_{nk}$; similarly, let b_k , c_k , $\cdots$

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