NOTE ON A THEOREM OF MACKEY

By L. H. Loomis

1. Let G be a locally compact group and let μ denote left-invariant Haar measure on G. We consider two families of bounded linear operators on $L^2(G;\mu)$, the unitary transformations $\{U_\sigma\}$ defined by left translations ($[U_\sigma f](s) = f(\sigma^{-1}s)$), and the projections $\{P_A\}$ defined by multiplication by the characteristic functions φ_A of Baire sets ($[P_A f](s) = \varphi_A(s)f(s)$). These two families of transformations are evidently related by: $U_\rho P_A f = P_{\rho A} U_\rho f$. Mackey [2] has proved a theorem which is roughly to the effect that if an abstract Hilbert space H admits a similar pair of families of unitary and projection operators, similarly related, then H must in fact be essentially $L^2(G)$ and the two families of operators must be essentially those described above. Mackey's proof makes use of the theory of unitary equivalence of self-adjoint operators and involves rather complicated measure-theoretic preliminaries. We present in this note a new approach to the problem, which allows us to prove Mackey's theorem directly and to strengthen it by dropping his hypothesis of separability. We should remark, however, that Mackey has subsequently applied his method in more general situations, whereas it is not clear that the present device can be similarly extended.

In the statement of the theorem below, a Baire set is understood to be any set of the Boolean σ-ring generated by the compact Baire sets. See [1; 24 and 220] for standard definitions.

Theorem. *Let $\rho \to U_\rho$ be a strongly continuous representation of a locally compact group G by unitary operators on a Hilbert space H, and let $A \to P_A$ be a non-zero Boolean σ-homomorphism from the Baire sets of G onto an Abelian family of projections on H, such that*

$$U_\rho P_A = P_{\rho A} U_\rho \tag{1}$$

for every $\rho \in G$ and every Baire set A. Suppose first that H is irreducible under the combined families $\{U_\rho\}$ and $\{P_A\}$. Then there exists a unique unitary mapping of $L^2(G)$ onto H such that U_ρ corresponds to left translation through ρ, and P_A corresponds to multiplication by the characteristic function φ_A of A. That is

$$[T^{-1}U_\rho Tf](s) = f(\rho^{-1}s)$$

$$[T^{-1}P_A Tf](s) = \varphi_A(s)f(s).$$

In general H is a direct sum of subspaces each of which is identifiable with $L^2(G)$ in the above way.

Received February 21, 1952.