PROPERTIES OF REGULAR RINGS

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1. **Introduction.** A ring R is *regular* if for each element $a \varepsilon R$ there exists an $x \varepsilon R$ with $axa = a$. A ring R is *biregular* if each principal ideal (the word ideal shall mean two-sided ideal unless explicitly stated otherwise) is generated by an idempotent in the center of R. A ring R is *strongly regular* if for each element $a \varepsilon R$ there exists an $x \varepsilon R$ with $a^2x = a$. Both regularity (introduced by von Neumann [8]) and biregularity (introduced by Arens and Kaplansky [1]) are generalizations of the classical semi-simple ring concept of Wedderburn, Artin, and others. However, the two generalizations are independent. Strong regularity (also due to Arens and Kaplansky [1]) implies both regularity and biregularity.

In the present paper we give analogues, for the above classes of rings, to the following known properties of a semi-simple ring R.

(I) *Every ideal is a principal ideal, having a unique idempotent generator in the center of R.*

(II) *R is isomorphic to a direct sum of its minimal ideals.*

(III) *The ideals of R form a Boolean algebra.*

(IV) *Every ideal is the meet of all maximal ideals containing it.*

Of particular interest is the following result; in a biregular ring the assumption of a non-zero annihilator for each maximal ideal implies that the ideals of the ring form a Boolean algebra, and the ring is isomorphic to the discrete direct sum of its minimal ideals.

In an arbitrary ring R, an ideal P is said to be *prime* if $A \cdot B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, where A and B are ideals in R. An ideal P is said to be *completely prime* if $ab \varepsilon P$ implies $a \varepsilon P$ or $b \varepsilon P$. An ideal A is said to be *indecomposable* if there do not exist ideals A' and A'' properly containing A, and such that $A' + A'' = R$, $A' \cap A'' = A$. We give the relations between the sets of prime, completely prime, maximal, and indecomposable ideals for the classes of rings considered.

2. **Regular rings.** The following characterization of a regular ring (due to von Neumann [8; Lemma 5]) provides an analogue to property I.

Theorem 1. *A ring R is regular if and only if for each $a \varepsilon R$ there exists an idempotent $e \varepsilon R$ such that a and e generate the same principal right ideal.*

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