THE EXISTENCE OF INVARIANT SUBSPACES

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Introduction. Let B be a Banach space and T a bounded linear operator on B. By a "non-trivial invariant subspace" of B with respect to T we shall mean a closed subspace C of B, $C \neq B$ and $C \neq \{0\}$, such that if $x \in C$ also $Tx \in C$. It is not known at present whether every bounded linear operator possesses at least one non-trivial invariant subspace. By a theorem of R. Godement [4; 136, Theorem J] this property holds for any linear and isometric operator on an arbitrary Banach space. It follows at once that the property also holds for any operator T with a bounded inverse for which $\| T^n \|$ is uniformly bounded for $n = 0, \pm 1, \pm 2, \cdots$, since under this restriction the space can be given an equivalent norm under which T is isometric. In this paper we shall consider the existence problem for invariant subspaces of operators T for which we assume that the sequence $\| T^n \|$, $n = 0, \pm 1, \pm 2, \cdots$ does not grow too rapidly.

Let $\{\rho_n\}$, $n = 0, \pm 1, \cdots$ be a sequence of positive numbers. We shall say that this sequence obeys condition (1) if it is majorized by a sequence $\{d_n\}$ in the sense that $\rho_n \leq d_n$ for all n, where $d_{-n} = d_n$, $d_n \geq 1$, all n, $\sum_0^\infty (\log d_n)/(1 + n^2) < \infty$, d_n is non-decreasing as $| n |$ increases and $(\log d_n)/n$ decreases as $| n |$ increases.

It is clear that if $\rho_n = O(e^{|n|^\alpha})$ for some α where $0 < \alpha < 1$, then $\{\rho_n\}$ satisfies (1). On the other hand, if $\rho_n \geq e^{n/\log n}$, $n \geq N_0$, then (1) fails for $\{\rho_n\}$.

Suppose $\rho_n = \| T^n \|$ for some bounded operator T. Then if $\{\rho_n\}$ satisfies (1), we may conclude, first, that $\| T^n \| \geq 1$ for all n, and secondly, that the spectrum of T lies on the unit circle. For suppose that some λ is in the spectrum of T and $| \lambda | > 1$. Then the spectral radius of T, $r(T)$, exceeds 1. Since $r(T)$ is $\lim_{n=\infty} \| T^n \|^{1/n}$, there is a number R greater than 1 such that $\| T^n \| > R^n$ for large n. Hence if $d_n \geq \rho_n$, we have that $\log d_n > n \log R$ for large n and so $\sum_0^\infty (\log d_n/1 + n^2) = \infty$. On the other hand, if $| \lambda | < 1$ and λ is in the spectrum of T, then $| 1/\lambda | > 1$ and $1/\lambda$ is in the spectrum of T^{-1}, again denying (1). Finally, if $\| T^m \| < 1$ and $m > 0$, we have for all positive k that $\| T^{mk} \| \leq \| T^m \|^k$ and hence $r(T) = \lim_{k=\infty} \| T^{mk} \|^{1/mk} < 1$, which is impossible by the preceding, and similarly for any negative m it is impossible that $\| T^m \| < 1$.

In §2 we shall prove Theorem 2 which states that if for an operator T on an arbitrary Banach space the sequence $\{\| T^m \|\}$ obeys condition (1) and if the spectrum of T does not reduce to a single point, then T possesses a non-trivial invariant subspace.

If $\| T^n \|$ does not grow more rapidly than a polynomial in n, we can drop the hypothesis that the spectrum of T contains at least two points. We have thus:

Received February 21, 1952; in revised form August 20, 1952.