THE KERNEL FUNCTION IN THE GEOMETRY OF MATRICES

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1. **Introduction.** Let $Z^{(m,n)} = Z = (z_{jk})$ $(1 \leq j \leq m; 1 \leq k \leq n)$ be a matrix of m rows and n columns whose elements are complex numbers; $Z^{*\prime}$ the conjugate transpose of Z and I the identity matrix. We consider the set of points such that $ZZ^{*\prime} < I$, consisting of all points Z with mn complex coordinates $(z_{11}, z_{12}, \cdots, z_{21}, \cdots, z_{mn})$ for which the hermitian quadratic form, $\sum_{j,k=1}^{m} (\delta_{jk} - \sum_{i=1}^{n} z_{ji} z_{ki}^{*}) u_j u_k^{*}$, in the auxiliary variables $(u_1, \cdots, u_m)$ is positive definite [8]. The symbol "Z" stands for both the matrix Z and the point Z since it is always clear from the context which is meant. By defining a neighborhood of a point Z, for example, as the set of points Y such that $| y_{jk} - z_{jk} | < \epsilon$ $(1 \leq j \leq m; 1 \leq k \leq n)$, it can be shown that the set of points Z such that $ZZ^{*\prime} < I$ forms a convex domain D, embedded in $2mn$-dimensional Euclidean space. If Z is a symmetric matrix $(Z = Z')$, the corresponding set of points $Z = (z_{11}, \cdots, z_{1n}, z_{22}, \cdots, z_{nn})$ lies in $n(n+1)$-dimensional Euclidean space, and, if Z is skew-symmetric $(Z = -Z')$, in $n(n-1)$-dimensional Euclidean space. These three domains form part of a set of six irreducible domains, possessing the property that all other bounded simple symmetric analytic spaces can be derived from them by analytical mappings and topological products [8, 9].

We construct the kernel function of the three domains mentioned above by analytic methods introduced by Bergman in the theory of functions of one and several complex variables [3]. Let L^2 be the class of analytic functions $f(Z) = f(z_{11}, \cdots, z_{mn})$, with finite norm, $N(f) = [\int \cdot_D \cdot \int | f(Z) |^2 \, dV]^{\frac{1}{2}}$, where dV is the Euclidean volume element. For the class L^2 we find the solution of a certain extremal problem and by means of a result due to Bergman show the connection with the kernel function of the domain. The kernel function turns out to be

$$K(T, Z^{*\prime}) = \frac{1}{V[\det (I - TZ^{*\prime})]^{p}}, \tag{1.1}$$

where $p = m + n$ for rectangular matrices, $n + 1$ for symmetric and $n - 1$ for skew-symmetric and V is the Euclidean volume of the domain D; for rectangular matrices, for example [6],

$$V = \frac{\pi^{mn} \prod_{j=1}^{m-1} j! \prod_{k=1}^{n-1} k!}{\prod_{i=1}^{m+n-1} i!}. \tag{1.2}$$

The kernel function of the domain D has all the usual properties, including the reproducing property (compare (2.6)).

Received March 8, 1952.