DOUBLE STURM-LIOUVILLE EXPANSIONS

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Given the differential equation

$$u''(x) - q(x)u(x) + \lambda u(x) = 0 \qquad 0 \leq x \leq \pi, \lambda \geq 0 \tag{1}$$

with $q(x)$ continuous and non-negative and the boundary conditions

$$u'(0) - hu(0) = 0 \qquad u'(\pi) + Hu(\pi) = 0 \qquad h, H, \text{ real}, \tag{2}$$

then it is well known that there exists a sequence of eigenvalues $\lambda_n \geq 0$ and corresponding eigenfunctions $\varphi_n(x)$ which satisfy (1) and (2). Concerning $q(x)$, we shall also assume that it is of bounded variation on $[0, \pi]$. The functions $\varphi_n(x)$ are determined to within a constant factor and form a complete orthogonal set with respect to the class of Lebesgue integrable functions on the closed interval $[0, \pi]$. We shall henceforth suppose that the functions $\varphi_n(x)$ are also normalized, that is $\int_0^\pi | \varphi_n(x) |^2 \, dx = 1$ and shall call these normalized functions the Sturm-Liouville orthonormal system.

Haar [1] has shown that if a function f which is integrable over $[0, \pi]$ is expanded in a Fourier series of Sturm-Liouville functions, then this series is uniformly equiconvergent on $[0, \pi]$, and hence uniformly equisummable by any regular method, with the Fourier cosine series of f. By two series being equiconvergent we mean, of course, that the sequence obtained by subtracting the n-th partial sum of one from the n-th partial sum of the other converges to zero. The importance of Haar's result is clear, for it enables us to refer questions concerning the representation of a function by its Sturm-Liouville series to questions concerning representation by trigonometric Fourier series. In the two-dimensional case Mitchell [3] has shown equiconvergence between Sturm-Liouville and cosine-cosine developments for functions which are of bounded variation in some suitable sense and has obtained equisummability $(C,1,1)$ almost everywhere for functions which are integrable. It is the purpose of this paper to show that if we use restricted convergence and summability rather that the usual Pringsheim method of passage to the limit, then in the convergence case we get results valid for functions which are merely assumed to be integrable, while in the summability case we can replace "almost everywhere" by "everywhere" for functions in L_p, $p > 1$. These results are stated in Theorems 1 and 2 below.

A double sequence S_{MN} is said to converge *restrictedly* to a limit l if for any fixed positive real constants A and B it is true that for every $\varepsilon > 0$ positive integers M_0 and N_0 exist with the property that whenever $M > M_0$, $N > N_0$

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