BOUNDS FOR THE CHARACTERISTIC ROOTS OF MATRICES WITH REAL ELEMENTS

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In this paper a theorem will be proved which in the case of square matrices with real elements will often give better bounds for the characteristic roots than Brauer's Theorem 35 (see preceding paper).

Let $A = (a_{\kappa\lambda})$ be a square matrix of order n with real elements. Using Brauer's notation, we let

$$P_\kappa = \sum_{\substack{\nu=1 \\ \nu\neq\kappa}}^{n} | a_{\kappa\nu} | \qquad (\kappa = 1, 2, \cdots, n),$$

and for each given κ and λ we denote the sum of the positive terms in

$$(1) \qquad S_{\kappa\lambda} = \sum_{\substack{\nu=1 \\ \nu\neq\kappa,\lambda}}^{n} a_{\kappa\nu}a_{\lambda\nu}$$

by $U_{\kappa\lambda}$, the sum of the negative terms by $V_{\kappa\lambda}$, and maximum $(U_{\kappa\lambda}, | V_{\kappa\lambda} |)$ by $W_{\kappa\lambda}$.

We set

$$P'_{\kappa\lambda} = | a_{\kappa\lambda} | P_\lambda + | a_{\lambda\kappa} | (P_\kappa - | a_{\kappa\lambda} |) + W_{\kappa\lambda} + \sum_{\nu<\mu} (| a_{\kappa\nu}a_{\lambda\mu} | + | a_{\lambda\nu}a_{\kappa\mu} |),$$

where $\kappa, \lambda, \nu, \mu$ run from 1 to n and where $\kappa \neq \lambda$; $\mu \neq \kappa, \lambda$; $\nu \neq \kappa, \lambda$.

Theorem. *Each characteristic root ω of A lies in the interior or on the boundary of at least one of the $n(n-1)/2$ ovals of Cassini*

$$| z - a_{\kappa\kappa} | \cdot | z - a_{\lambda\lambda} | \leq P'_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \cdots, n; \kappa \neq \lambda).$$

Proof. Let ω be a characteristic root of A, then $\bar{\omega}$ is also a root.

Since ω is a characteristic root of A, the system of linear equations

$$(2) \qquad \sum_{\nu=1}^{n} a_{\kappa\nu}x_\nu = \omega x_\kappa \qquad (\kappa = 1, 2, \cdots, n)$$

has a non-trivial solution $(\zeta_1, \zeta_2, \cdots, \zeta_n)$. Furthermore we have $(\bar{\zeta}_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_n)$ as a solution to the system

$$(3) \qquad \sum_{\nu=1}^{n} a_{\kappa\nu}x_\nu = \bar{\omega} x_\kappa \qquad (\kappa = 1, 2, \cdots, n).$$

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