LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX. V.

By Alfred Brauer

In part II of this paper [1; Theorem 11] it was proved that each characteristic root of a square matrix $A = (a_{\kappa\lambda})$ of order n lies in the interior or on the boundary of at least one of the $n(n-1)/2$ ovals of Cassini

$$|z - a_{\kappa\kappa}|\,|z - a_{\lambda\lambda}| \leq P_\kappa P_\lambda \qquad (\kappa, \lambda = 1, 2, \cdots, n;\ \kappa \neq \lambda) \tag{101}$$

where

$$P_\kappa = \sum_{\substack{\nu=1 \\ \nu\neq\kappa}}^{n} |a_{\kappa\nu}| \qquad (\kappa = 1, 2, \cdots, n).$$

We set

$$\begin{aligned} P_{\kappa\lambda} = {} & |a_{\kappa\lambda}|\, P_\lambda + |a_{\lambda\kappa}|\,(P_\kappa - |a_{\kappa\lambda}|) + \sum_{\nu} |a_{\kappa\nu}a_{\lambda\nu}| \\ & + \sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu} + a_{\kappa\mu}a_{\lambda\nu}| \end{aligned} \tag{102}$$

where κ, λ, μ, and ν run from 1 to n and where $\kappa \neq \lambda$, $\mu \neq \kappa, \lambda$; $\nu \neq \kappa, \lambda$. It will be shown in this paper that the ovals (101) can be replaced by the ovals

$$|z - a_{\kappa\kappa}|\,|z - a_{\lambda\lambda}| \leq P_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \cdots, n;\ \kappa \neq \lambda). \tag{103}$$

It is obvious that

$$P_{\kappa\lambda} \leq P_\kappa P_\lambda\,. \tag{104}$$

We have the equality sign in (103) if and only if

$$\sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu} + a_{\kappa\mu}a_{\lambda\nu}| = \sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu}| + \sum_{\nu<\mu} |a_{\kappa\mu}a_{\lambda\nu}|.$$

Using this result instead of Theorem 11 we can improve Theorems 12–19, and 22. In particular, the following theorem will be obtained.

Assume that

$$|a_{\kappa\kappa}a_{\lambda\lambda}| > P_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \cdots, n;\ \kappa \neq \lambda).$$

Then the determinant of A does not vanish. If, moreover, $a_{\kappa\kappa} > 0$ for $\kappa = 1, 2, \cdots, n$, and if the characteristic equation has real coefficients, then the determinant of A is positive.

This improves Theorem 16. While the assumptions of Theorem 16 imply that the inequality

$$|a_{\kappa\kappa}| > P_\kappa \tag{105}$$

holds for all but one κ, none of these inequalities is required for the new result.

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