CONGRUENCES FOR THE MÉNAGE POLYNOMIALS

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1. **Introduction.** In a recent paper [2], Riordan showed that the generalized ménage numbers $u_{n,r}$ satisfy the recurrence

$$n! = \sum_{k=0}^{n} \binom{2n}{k}(1 - t)^k U_{n-k}(t), \tag{1.1}$$

where

$$U_n = U_n(t) = \sum_{r=0}^{n} u_{n,r} t^r. \tag{1.2}$$

Making use of (1.1), he proved that U_n satisfies the congruence

$$U_{p^2+n} \equiv (t^{p^2} - 1) U_n \pmod p, \tag{1.3}$$

where p is a prime > 2.

In the present note we prove the stronger result

$$U_{m^2+n} \equiv (t - 1)^{m^2} U_n \pmod m, \tag{1.4}$$

where m is an arbitrary integer ≥ 2. In place of (1.1) we make use of the formulas [1; 121]

$$U_n = nW_{n-1} + 2(t - 1)^n = W_n - (t - 1)^2 W_{n-2} \qquad (n > 0), \tag{1.5}$$

and

$$W_n = nW_{n-1} + (t - 1)^2 W_{n-2} + 2(t - 1)^n \qquad (n > 0), \tag{1.6}$$

which is a consequence of (1.5). Here $W_n = W_n(t)$ is a polynomial which may be defined by

$$W_n(t) = \sum_{k=0}^{n} \binom{2n - k + 1}{k} (n - k)!(t - 1)^k.$$

(Following a suggestion of Riordan, we write $V_n(t)$, $W_n(t)$ in place of the $H_n(t)$, $I_n(t)$ of [1]. This is in agreement with the notation of [2].)

2. We first establish the congruence

$$W_{m+k-1} - (t - 1)^{2k} W_{m-k-1} \equiv 2(t - 1)^m W_{k-1}, \tag{2.1}$$

for $1 \leq k \leq m - 1$; all congruences are modulo m (except as noted).

If we take $m = n$ in (1.6) we get

$$W_m \equiv (t - 1)^2 W_{m-2} + 2(t - 1)^m \qquad (\text{or } U_m \equiv 2(t - 1)^m); \tag{2.2}$$

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