INFINITE LIE ALGEBRAS

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1. **Introduction.** The object of this paper is to study Lie algebras whose underlying vector spaces need not be finite dimensional. Under suitable restrictions on the Lie algebras we will prove extensions of the theorems of Engel, Lie, and Levi as follows: If every element of a Lie algebra is nilpotent then the algebra is nilpotent. If a Lie algebra is solvable then its derived algebra is nilpotent. A Lie algebra is the direct sum of its radical and a semisimple algebra. As special results we will show that if in a Lie algebra every proper subalgebra can be embedded in a maximal proper subalgebra which is an ideal then the algebra is nilpotent; we will also prove an extension of the author's tower theorem.

The restrictions we impose are the following.

(1) *The Lie algebra L has a nil radical N such that $N^k = 0$ and L/N is finite dimensional.*

Here, as usual, the nil radical refers to the join of all the nilpotent ideals of L and we assume that $N^k = 0$ for some integer k . It would actually be equivalent to assume only the existence of a nilpotent ideal M such that L/M is finite dimensional, since then the join of all the nilpotent ideals of L is the join of only a finite number of them and hence is a nilpotent ideal.

(2) *L is locally finite; that is, any finite number of elements of L can be embedded in a finite subalgebra of L.*

Sometimes we will need the following stronger condition.

(2′) *L is uniformly locally finite; that is, the dimension of the subalgebra $\{l_1, \cdots, l_n\}$ of L generated by $l_1 \cdots, l_n$ is less than or equal to $\phi(n)$, where $\phi(n)$ depends only on n.*

2. Extensions of the theorems of Engel, Lie and Levi.

Theorem 1. *If L is a Lie algebra with nil radical N such that L/N is finite dimensional, and if the adjoint mapping l^* of every element l of L is nilpotent then L is nilpotent.*

Proof. We must show that $N = L$. If $N \neq L$ then, by Engel's theorem, L/N is nilpotent and hence has a non-zero center. Consequently there is an

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