THE MINIMA OF SOME NON-HOMOGENEOUS FUNCTIONS OF TWO VARIABLES

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It is known that if $f(x, y)$ is a function of two variables x, y, there are results given by the

Theorem. *Let x_0, y_0 be any real numbers. Then for certain functions $f(x, y)$, there exist numbers x, y such that*

$$x \equiv x_0 \pmod 1, \qquad y \equiv y_0 \pmod 1, \tag{1}$$

and

$$|f(x, y)| \le k \max(|f(1, 0)|, \ |f(0, 1)|, \ |f(1, 1)|, \ |f(1, -1)|), \tag{2}$$

where k is a number to be defined and depending only upon $f(x, y)$.

Take $f(x, y)$ to be a binary form of degree n and $k = 2^{-n}$. When $n = 2$ and $f(x, y)$ is an indefinite form, Barnes [2] has proved the theorem in the sharper form with $|f(1, 1)|$, $|f(1, -1)|$ replaced by $\min |f(1, \pm 1)|$. When $n = 3$, the theorem has been proved by Bambah [1] when the discriminant of $f(x, y)$ is positive, and by Chalk [3] when the discriminant is negative. In studying Chalk's proof, it seemed to me that his result was of such a character that it should be capable of proof without any detailed calculation, and that it must be a particular case of a more general theorem. By using some of Chalk's ideas and with little formal calculation, I prove this.

Write $X = px + qy$, $Y = rx + sy$, where p, q, r, s are real numbers and $ps - qr \neq 0$. Special cases of my result are given by (2) with

$$f(x, y) = X^n + Y^n, \qquad k = 2^{-n} \qquad (n > 1),$$

where n is the quotient of two positive odd integers. Chalk's result is the case $n = 3$. Another instance is

$$f(x, y) = X^3 + Y^3 + l(X + Y), \qquad k = \tfrac{1}{2} \qquad (l \ge 0).$$

This means that $f(x, y)$ can be written as

$$f(x, y) = (ex + fy)(ax^2 + bxy + cy^2 + 1),$$

where a, b, c, e, f are all real, and the binary quadratic term is positive definite.

Finally we take

$$f(x, y) = Ye^{X^n}, \qquad k = \tfrac{1}{2} \qquad (1 < n \le 2),$$

where n is the quotient of an even integer by an odd one.

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