APPROXIMATELY CONVEX FUNCTIONS

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1. **Introduction.** A novel generalization of convex function has been introduced by D. H. Hyers and S. M. Ulam [2]. A real valued function f defined on an n-dimensional convex set S is said to be ϵ-approximately convex, or more briefly, ϵ-convex, provided for every x,y in S and for every λ, $0 \leq \lambda \leq 1$, it satisfies the inequality

$$(1) \qquad f(\lambda x + (1 - \lambda)y) \leq \epsilon + \lambda f(x) + (1 - \lambda)f(y).$$

Here and henceforth, the letters x, y, z and only these, with or without subscripts, will represent points or vectors in n-dimensional Euclidean space E. The positive number ϵ is fixed throughout the entire paper, and, indeed, could be replaced by 1 without any loss in generality. If ϵ were zero, this would be ordinary convexity. λ is always a number between 0 and 1.

Hyers and Ulam proved a theorem which amounts to the following: If f in continuous and ϵ-convex in a convex domain S, there exists a convex function g such that in S, $g(x) \leq f(x) \leq g(x) + k_n\epsilon$, where $k_n = 1 + (n - 1)(n + 2)/2(n + 1)$. The constant k_n is the smallest possible one for $n = 1, 2$, but not beyond, as will appear later. In fact, k_n is of too great an order of magnitude for large n. In the following, we shall prove this theorem in a different manner, extending it to upper or lower semicontinuous functions, and obtain an improved value of k for $n \geq 3$. A slightly weaker theorem will be obtained for general discontinuous functions. In addition a number of miscellaneous properties of ϵ-convex functions will be obtained.

2. **Continuity properties of ϵ-convex functions.** In the following, S will be a convex open set in E, not necessarily bounded.

Theorem 1. *If f is ϵ-convex on S, it is bounded above on each compact subset of S, and bounded below on each bounded subset of S.*

The proof does not differ materially from that of the corresponding theorem for convex functions, and being very simple, will be omitted.

Theorem 2. *The oscillation of f at any point in S does not exceed ϵ.*

For simplicity, consider the oscillation at 0, where we may assume that $\liminf_{x\to 0} f(x) = 0$ without loss of generality. (By $\liminf_{x\to 0} f(x)$ we mean $\lim \inf_{R\to 0;\, |x|<R} f(x)$, and similarly for $\limsup_{x\to 0} f(x)$.) If the theorem is false, there exist sequences $\{x_n\}$ and $\{y_n\}$ tending to 0 and such that $\lim f(x_n) = 0$, $\lim f(y_n) = \alpha > \epsilon$. Let K be the sphere $|x| = a$, where $|x|$ is the length of

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