A NOTE ON THE HEAT EQUATION

BY I. I. HIRSCHMAN, JR.

1. Let $u(x, t)$ satisfy the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

for $-\infty < x < \infty$, $-\infty < t \leq c$. We may think of $u(x, t)$ as the temperature of an infinite insulated bar extended along the x axis, $u(x, t)$ being the temperature of the rod at the point x and the time t. Since the beginning of time ($t = -\infty$) the heat in the bar has undergone a process of diffusion, flowing from regions of high temperature to regions of lower temperature. Because at any finite moment this mixing process has already gone on for an infinite length of time, it is natural to conjecture that $u(x, t)$ must be constant. However this is not true without some additional hypothesis as is shown by the example

$$u(x, t) = e^{-t} \sin x. \tag{2}$$

Appel [1] has proved that if we assume

$$0 \leq u(x, t) \leq M \qquad (-\infty < x < \infty, -\infty < t \leq c), \tag{3}$$

where M is a constant independent of x and t, then $u(x, t)$ must be constant. (Appel proved slightly less; however, his argument can, using the results of [6], be made to yield the above result.) In the present paper we shall show that $u(x, t)$ is constant under the weaker assumptions

$$0 \leq u(x, t) \qquad (-\infty < x < \infty, -\infty < t \leq c), \tag{4}$$

$$\liminf_{r \to \infty} r^{-1} \log M(r) \leq 0, \tag{4'}$$

where $M(r) \geq$ l.u.b. $u(x, t)$ for $|x| \leq r$, $t = c$. The functions

$$u(x, t) = e^{a^2 t} \cosh ax$$

show that if $\liminf_{r\to\infty} r^{-1} \log M(r) > 0$ then $u(x, t)$ need not be constant.

2. Since it is no more difficult, we shall establish our theorem in m-dimensional space E_m. Points in E_m will be denoted by Greek letters, for example, $\xi = (x_1, x_2, \cdots, x_m)$ and $\eta = (y_1, y_2, \cdots, y_m)$. Let R be a set of points in the space whose coordinates are (ξ, t), where $\xi \in E_m$ and $-\infty < t < \infty$. We

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