ELEMENTARY DIVISORS OF CERTAIN MATRICES

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In a recent paper Flanders [1] proved that the elementary divisors of AB and of BA are identical except for those corresponding to the characteristic root zero. He also showed that for N nilpotent and such that $NA = 0$, the matrices $AB + N$ and AB have the same elementary divisors except for those corresponding to the characteristic root zero.

In this note we establish a similar theorem and show how both the above results are obtained from it.

THEOREM 1. *Let P and Q be two $n \times n$ matrices such that $P^s = QP^{s-1}$ and $Q^t = PQ^{t-1}$. Then P and Q have the same elementary divisors except for those corresponding to the chracteristic root zero.*

We observe that $P^{s-1}(\lambda I - P) = (\lambda P^{s-1} - P^s) = (\lambda I - Q)P^{s-1}$ and, consequently,

$$P^{s-1}(\lambda I - P)^k = (\lambda I - Q)^k P^{s-1}, \tag{1}$$

for all positive integers k. Similarly,

$$Q^{t-1}(\lambda I - Q)^k = (\lambda I - P)^k Q^{t-1}. \tag{2}$$

Suppose that $\lambda \neq 0$ is a characteristic root of P and that $\xi \neq 0$ is a vector such that

$$(\lambda I - P)^k \xi = 0. \tag{3}$$

Then

$$P^{s-1}(\lambda I - P)^k \xi = (\lambda I - Q)^k P^{s-1} \xi = 0. \tag{4}$$

If $P^u \xi = 0$, then from (3) $\lambda^k P^{u-1} \xi = 0$ and hence $P^{u-1}\xi = 0$. Since $\xi \neq 0$ it follows that $P^{s-1}\xi \neq 0$. From (4) it follows that λ is a characteristic root of Q and that the nullity of $(\lambda I - P)^k$ does not exceed the nullity of $(\lambda I - Q)^k$. If $\eta \neq 0$ is any vector such that $(\lambda I - Q)^k \eta = 0$, then in a similar manner it follows that $(\lambda I - P)^k Q^{t-1} \eta = 0$, where $Q^{t-1}\eta \neq 0$. Hence the nullity of $(\lambda I - Q)^k$ does not exceed the nullity of $(\lambda I - P)^k$. Therefore, for $\lambda \neq 0$, $(\lambda I - P)^k$ and $(\lambda I - Q)^k$ have the same nullity for every positive integer k. Thus the elementary divisors associated with the characteristic root $\lambda \neq 0$ are identical for the two matrices P and Q.

Let $N = P - Q$ be nilpotent of index t and such that $NQ = 0$, then $(P - Q)^s = (P - Q)P^{s-1}$. Hence $(P - Q)P^{t-1} = 0$ or $P^t = QP^{t-1}$ and from $NQ = (P - Q)Q = 0$, $Q^2 = PQ$. Thus we have the

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