CONVEXITY AND STARLIKENESS OF ANALYTIC FUNCTIONS

By W. C. Royster

1. **Introduction.** Necessary and sufficient conditions for an analytic function $f(z)$ to map a circle $|z| = r$ onto a curve which is convex, or starlike with respect to a given point, are well known [3; 105]. In this paper we obtain generalizations of these conditions for certain other elementary types of curves in the z-plane and obtain some consequences of these generalizations.

Let $\Gamma = \Gamma(t)$ denote a directed curve $z(t) = x(t) + iy(t)$, where for $t_0 \leq t \leq t_1$, the real functions $x(t)$ and $y(t)$ are twice differentiable and $|z'(t)| \neq 0$. The direction of the curve will be that generated by increasing t. We shall also consider one parameter families of such curves, $\Gamma = \Gamma(t, u): z = x(t, u) + iy(t, u)$, where for each fixed u in $u_0 \leq u \leq u_1$, $\Gamma(t, u)$ has the above mentioned properties for $t_0(u) \leq t \leq t_1(u)$.

Definition 1. Let C be the image of Γ under $w = f(z)$, and ψ be the angle made by the tangent line to C with the positive real axis. Then $f(z)$ is said to be *convex* on the directed curve Γ if

$$\frac{d\psi}{dt} \geq 0 \tag{1.1}$$

for $t_0 \leq t \leq t_1$.

It will be convenient to use the notation $f(z)$ for convex functions and $F(z)$ for starlike functions.

Definition 2. Let $\phi = \arg (F(z) - w_0)$, and suppose that on Γ, $F(z) \neq w_0$. Then $F(z)$ is said to be *starlike* with respect to w_0 on the directed curve Γ if

$$\frac{d\phi}{dt} \geq 0 \tag{1.2}$$

for $t_0 \leq t \leq t_1$.

In case $w_0 = 0$, we shall merely say "$F(z)$ is starlike on Γ." If $f(z)$ is defined in some region R and the curve Γ extends beyond this region, the statement "$f(z)$ is convex on Γ" will mean, $f(z)$ is convex on that portion of Γ lying in R.

2. **The fundamental inequalities.** First consider the convex case. If ψ_1 is the angle which a chord of the curve C makes with the positive real axis, then

$$\psi_1 = \arg (f(z + \Delta z) - f(z)) = \arg \frac{f(z + \Delta z) - f(z)}{\Delta z} + \arg \frac{\Delta z}{\Delta t},$$

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