ON QUASI-COMPACT MAPPINGS

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If X and Y are topological spaces, a mapping $f(X) = Y$ of X onto Y is *quasi-compact* [1] provided the image of every open inverse set in X is open in Y, that is, every open set U in X satisfying

$$U = f^{-1}f(U)$$

has as image an open set $f(U)$ in Y. Precisely the same concept is obtained if we substitute the word "closed" for "open" throughout this definition since, for inverse sets, the image of the complement is the complement of the image. Thus it is clear that all closed mappings as well as all open mappings are quasi-compact. In addition we note next some other classes of mappings which are included among the quasi-compact mappings. For our purposes any subset of X which maps onto all of Y under f will be called a *cross section* of f.

Theorem 1. *If a mapping $f(X) = Y$ is quasi-compact on some cross section A of X, it is quasi-compact on X.*

For let U be any open inverse set for f. Then $A \cdot U$ is an inverse set for the mapping $f|A$ and is open in A. Hence $f(A \cdot U)$ is open in Y. But $f(A \cdot U) = f(U)$ since A is a cross section and $U = f^{-1}f(U)$.

Corollary 1. *All retractions are quasi-compact.*

Corollary 2. *Any mapping which is closed* (*or open*) *on some cross section is quasi-compact.*

Theorem 2. *Local connectedness is invariant under all quasi-compact mappings.*

This results at once from two statements of a more general character as follows

(i) If $f(X) = Y$ is any mapping, the inverse $f^{-1}(Q)$ of any component Q of any set U in Y is the union of a collection of components of $f^{-1}(U)$.

For if R is a component of $f^{-1}(U)$ intersecting $f^{-1}(Q)$, we have $f(R) \subset Q$ because $f(R)$ is connected, lies in U and intersects Q.

(ii) If X is locally connected and $f(X) = Y$ is a mapping, the inverse set $f^{-1}(Q)$ of any component Q of an open set U in Y is open in X.

For by (i), $f^{-1}(Q)$ is the union of a collection of components of the open set $f^{-1}(U)$, and each such component is open by local connectedness of X.

Now to prove the theorem, we have only to note that, assuming X locally connected and $f(X) = Y$ quasi-compact, if Q is any component of an open set U in Y, $f^{-1}(Q)$ is open in X by (ii) and thus $Q = ff^{-1}(Q)$ is open in Y by quasi-compactness. Thus Y is locally connected.

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