A GENERAL SET-SEPARATION THEOREM

By J. W. Ellis

1. **Introduction.** This note deals with a problem which arises frequently and in many different forms: Given two disjoint subsets of a set S, one of type A, the other of type B, when is it possible to enlarge each of these to a set of the same type in such a way that the two new sets are complementary in S?

Well-known results of this type are those of Stone [3], for ideals in a distributive lattice, and Tukey [4], for convex sets in a real linear space. These two theorems are corollaries to the main result of this paper; a further application, to a theorem on cones by Klee [2], is also given here.

2. **Notation and Postulates.** Throughout this discussion, S will be a set, 2^S the collection of all subsets of S; ϕ and ψ will denote functions on 2^S to 2^S. We shall use $\cup$ and $\cap$ for point-set union and intersection, respectively. The empty set will be designated by Λ; the set whose only member is the point p by $\{p\}$. For brevity, we will write $\phi(a,b)$ for $\phi(\{a\} \cup \{b\})$.

The following postulates will be used in the discussion, and we will refer to them by number. The first four may be satisfied by a function ϕ on 2^S to 2^S, the fifth by a pair of such functions.

(P1) $\phi(E) \supset E$, for all $E \subset S$.

(P2) $\phi^2 = \phi$; that is, $\phi(\phi(E)) = \phi(E)$, for all $E \subset S$.

(P3) $\phi(E) = \cup \{\phi(F) \mid F \text{ finite and } F \subset E\}$.

(P4) If $F \subset S$ is finite and $p \in S$, then

$$\phi(F \cup \{p\}) \subset \cup \{\phi(a,p) \mid a \in \phi(F)\}.$$

(P5) If $a \in \psi(b,p)$ and $c \in \phi(d,p)$, then $\phi(a,d) \cap \psi(b,c) \neq \Lambda$.

2.1. Definition. If ϕ is a function on 2^S to 2^S, then $A \subset S$ is a *ϕ-set* if and only if $\phi(A) = A$.

2.2. *Remarks.* There are many examples of functions which satisfy some of the above postulates. If $\mathfrak{L}$ is any family of subsets of S such that S is a member of $\mathfrak{L}$ and the intersection of an arbitrary collection of members of $\mathfrak{L}$ is again a member of $\mathfrak{L}$, we may define, for each $E \subset S$,

$$L(E) = \cap \{L \mid L \in \mathfrak{L}, L \supset E\}.$$

This function L is easily seen to satisfy (P1) and (P2), and if the family $\mathfrak{L}$

Received April 9, 1952; presented to the American Mathematical Society, November 23, 1951. The author is an Atomic Energy Commission Predoctoral Fellow. He wishes to thank Professor B. J. Pettis for the suggestion which led to this paper.