NOTE ON THE EXTENSION OF RECTANGLE FUNCTIONS

BY H. KOBER

1. **Introduction.** Let $\mathfrak{R}$ be the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ (a, b fixed), and let $\mathfrak{R}_\lambda (0 \leq \lambda < 1)$ be the set of rectangles $R \subset \mathfrak{R}$ for which the ratio of the shorter to the longer side is $\geq \lambda$ (λ fixed). All the rectangles occurring, denoted by R, r or ρ, are oriented, that is, have sides parallel to the axes; R is a closed, R^0 an open rectangle. Similarly I or i is a closed, I^0 or i^0 an open oriented linear interval.

Let $f^\lambda(R)$ be an interval function defined in Eu_2, the Euclidean space of two dimensions, for every rectangle $R \subset \mathfrak{R}_\lambda$, and write $\mathfrak{R}$ for $\mathfrak{R}_\lambda$ and $f(R)$ for $f^\lambda(R)$ when $\lambda = 0$. If there exists a completely additive function $\phi(B)$ which is defined for every Borel set $B \subset \mathfrak{R}$ and such that $\phi(R) = f^\lambda(R)$ for any $R \subset \mathfrak{R}_\lambda$, then $f^\lambda(R)$ is said to admit an extension; it is said to admit a β-extension if $\phi(B) = f^\lambda(R)$ whenever $R \subset \mathfrak{R}_\lambda$ and $R^0 \subset B \subset R$ [4; §1.6]. The extension of $f^\lambda(R)$ to $\phi(B)$ plays a considerable part in analysis [6; Chapter III]. Results that are of particular value for applications have been given by Reichelderfer and Ringenberg [3], Ringenberg [4], Rechard and Reichelderfer [2], and by Radó [1]. For example, the following results are cited.

A non-negative function $f(R)$ ($R \subset \mathfrak{R}$) *admits an extension if and only if it satisfies* (a) *the condition* D *below* [3] (*for the case* $0 \leq \lambda < 1$, *see* [4]) *or* (b) *the conditions* E *below* [2].

D. THE RADÓ-REICHELDERFER-RINGENBERG CONDITION. $f(r_1) + \cdots + f(r_m) \leq \sum f(R_j)$ *whenever m is finite,* $r_j r_k = 0$ *for* $j \neq k$ *and* $\sum r_i \subset \sum R_i \subset \mathfrak{R}$, *where the* R_j*'s are finite or infinite in number and may overlap.*

E. THE RECHARD-REICHELDERFER CONDITIONS.

(i) $f(R_1) + f(R_2) \leq f(R)$ *when* $R_1 + R_2 \subset R \subset \mathfrak{R}$ *and* $R_1 R_2 = 0$.

(ii) $f(R) \leq f(R_1) + f(R_2)$ *when* $R \subset R_1 + R_2 \subset \mathfrak{R}$.

(iii) *If* $R \subset \mathfrak{R}$, $R_\epsilon = \mathfrak{R} R_\epsilon^*$, *where* R_ϵ^* *is the rectangle containing R with sides parallel to those of R at a distance* $\epsilon (\epsilon > 0)$, *then* $f(R_\epsilon) \to f(R)$ *as* $\epsilon \to 0$.

Plainly E(iii) is a condition of continuity. A similar condition plays a part in Ringenberg's results [4]. It is easily shown, and probably known, that D is equivalent to

D′. (i) $f(r_1) + \cdots + f(r_m) \leq f(R)$, *where* $\sum r_i \subset R \subset \mathfrak{R}$, $r_j r_k = 0$ for $j \neq k$, *and* (ii) $f(r) \leq \sum f(R_j)$, *where* $r \subset \sum R_j \subset \mathfrak{R}$.

Evidently D′(i) is satisfied when $f(R)$ is non-negative and additive. Note that the actual results are more general than stated above.

2. In the present paper a *continuity condition* is introduced which, in the special case of a non-negative and additive function $f(R)$, turns out to be equivalent to E(iii) and by which, therefore, our Theorem 1(b) can be reduced to the Rechard-

Received September 17, 1951; in revised form, January 25, 1952.