FUNCTIONS PERIODIC MODULO EACH OF A SEQUENCE OF INTEGERS

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Introduction. It is our purpose to study the integral valued functions $f(x)$ which have the property that for a given sequence of integers $1 \leq m_1 < m_2 < \cdots$ (finite or infinite) there exist $\pi_1, \pi_2, \cdots$ such that

$$f(x + \pi_i) \equiv f(x) \pmod{m_i} \qquad (i = 1, 2, \cdots), \tag{1}$$

where x and π_i are either integers or vectors with integral components. In the following it will be important to distinguish among different types of periodicity.

Definition. The function $f(x)$ is called:

periodic of type I mod $\{m_i\}$ if equation (1) holds for all $x = 0, \pm 1, \pm 2, \cdots$;

periodic of type II mod $\{m_i\}$ if equation (1) holds for all $x > x_0$, $i = 1, 2, \cdots$;

periodic of type III mod $\{m_i\}$ if equation (1) holds for all $x > x_i$, $i = 1, 2, \cdots$.

(In case x stands for a vector, the vectors x_0, x_i may have components $-\infty$).

The study of functions periodic mod $\{m_i\}$ arises in several widely differing connections. One problem which has had very little satisfactory treatment is that of prime representing functions. We again distinguish between different types.

Definition. The function $f(x)$ is called:

prime representing of type I if $f(x) =$ prime, either for all $x = 0, \pm 1, \pm 2, \cdots$ or for all $x > x_0$, x_0 given;

prime representing of type II if $f(x) =$ prime for all sufficiently large x, $x > x_0$, x_0 not given.

Our experience with the irregularity of primes leads us to the intuitive conviction that there exist no nontrivial "elementary" prime representing functions (as trivial functions might be considered those assuming only a finite number of values all of which are primes). A reasonably rigorous determination of "elementary" seems to be difficult, since too large a class of functions, such as, say, the class of all analytic functions, would permit interpolation, and thus would certainly contain prime representing functions. The proof that a function is not prime representing of type I is usually given by the demonstration of a specific counterexample; as, for example, in the case of the Fermat numbers

$$f(x) = 2^{2^x} + 1, \qquad x_0 = 0; \qquad \text{counterexample: } x = 5.$$

As to proofs that a function is not prime representing of type II only the following, essentially trivial, methods seem to have been used:

(a) *Factorization.* That is, the demonstration of a relation

$$f(x) = g(x)h(x),$$

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