THE ITERATION OF POWER SERIES IN TWO VARIABLES

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1. **Introduction.** It was shown by Koenigs [5], [6] that if

$$f(x) = \sum_{k=1}^{\infty} a_k x^k \tag{1}$$

has a non-zero radius of convergence and $| a_1 |$ is less than 1 and not equal to zero, then for $| x |$ sufficiently small, there exists a function of two variables, $f(x, t)$ possessing the properties of a generalized iterate, which is to say

$$\begin{aligned} &f(x, 0) = x, \ f(x, n) = n\text{-th iterate of } x \text{ for } n \text{ a positive integer;} \\ &f(f(x, s), t) = f(x, s + t) \qquad (s, t \geq 0). \end{aligned} \tag{2}$$

This function has the elegant representation,

$$f(x, t) = \phi^{-1}[a_1^t \phi(x)], \tag{3}$$

where

$$\phi(x) = \lim_{n \to \infty} f^{(n)}(x)/a_1^n . \tag{4}$$

The function $\phi(x)$ is itself an interesting and important function, since it "linearizes" $f(x)$, that is,

$$\phi(f(x)) = a_1 \phi(x). \tag{5}$$

The problem of finding a $\phi(x)$ satisfying (5), given $f(x)$ has a long history going back to Abel. For further discussion and references, we refer to Hadamard [4] and to H. Töpfer [8]. (This last reference was furnished by the referee to whom we are indebted for this and many other helpful comments.)

Let us now turn our attention to power series in two variables. Let

$$f(x, y) = \sum a_{kl} x^k y^l, \qquad g(x, y) = \sum b_{kl} x^k y^l \qquad (k, l \geq 0, k + l \geq 1) \tag{6}$$

be convergent for $| x |$ and $| y |$ sufficiently small and let us make the assumption that the characteristic roots of the matrix of the coefficients of the linear terms,

$$A = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}, \tag{7}$$

are non-zero and less than unity in absolute value.

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