CONGRUENCES FOR THE COEFFICIENTS OF HYPERELLIPTIC AND RELATED FUNCTIONS

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1. **Introduction.** In a previous paper [2] the writer considered the coefficients of the Jacobi elliptic function

$$f(x) = \text{sn}\,(x, k^2) = \sum_{m=1}^{\infty} \frac{a_m x_m}{m!}, \tag{1.1}$$

where the rational number $l = k^2$ is integral (mod p), p an odd prime, and proved

$$\sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} a_p^{r-i} a_{m+i(p-1)} \equiv 0 \pmod{p^r} \tag{1.2}$$

for $m \geq r \geq 1$. The question was raised there whether like results hold in the hyperelliptic case, for example for the function $g(x) = \sum_1^\infty c_m x^m/m!$ satisfying

$$g'^2(x) = 1 + A_1 g(x) + \cdots + A_6 g^6(x), \tag{1.3}$$

where $A_1, \cdots, A_6$ are integral (mod p). The method of [2] apparently fails for the following reason. In the case of (1.1) we find that $D^{p-1}f$, where D denotes d/dx, is a polynomial in f of degree p, and by means of this and some previous results it is shown that

$$(D^p - a_p D)f = p \sum_{i=0}^{\infty} d_i f^i, \tag{1.4}$$

where the d_i are integral (mod p); (1.2) follows fairly easily from (1.4). Now for a function satisfying (1.3), we can only assert that $D^{p-1}g$ is a polynomial in g of degree $\leq 2p - 1$, and it does not seem possible to prove a result like (1.4).

The method of [2] can however be modified to apply to the hyperelliptic case and more generally to the class of functions $g(x) = \sum_1^\infty c_m x^m/m!$ such that the inverse function is of the form $\sum_1^\infty e_m x_m/m$, where the c_m, e_m are integral (mod p) and $c_1 = e_1 = 1$. (This is the class of functions for which a generalization of the Staudt-Clausen theorem has been proved [1]). We are however not able to prove (1.2) in the present case but only the weaker result

$$\sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} c_p^{r-i} c_{m+i(p-1)} \equiv 0 \pmod{p^s}, \tag{1.5}$$

where $s = [\frac{1}{2}(r + 1)]$. A formula similar to (1.5) also holds for the coefficients of $g^\lambda(x)$ (see (3.8) below).

In the next place, let

$$\frac{x}{g(x)} = \sum_{m=0}^{\infty} \frac{\beta_m x^m}{m!};$$

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