VARIATIONAL METHODS AND NON-OSCILLATION THEOREMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. **Introduction.** The purpose of this paper is to establish by variational methods for certain systems of ordinary differential equations a number of non-oscillation theorems analogous to those recently given by E. Hille [5] and A. Wintner [13], [14] for a single linear second order differential equation. Each of the theorems established will be seen to be equivalently a theorem on the non-existence of conjugate points for a fixed end point problem of Lagrange in the calculus of variations.

The system of differential equations which we consider is introduced in §2 and is composed of n linear second order differential equations together with m, $0 \leq m < n$, linear first order differential equations each with real coefficients; extensive use will be made of the fact that the equations considered are the accessory differential equations for a problem of Lagrange. In §3 two fundamental lemmas of a variational nature, which may be established with the aid of a result proved by W. T. Reid [11], are stated without proof and are used in §4 to establish a number of necessary and sufficient conditions for non-oscillation of the given system of differential equations for large x on an infinite interval, that is, for the non-existence of conjugate points for such x. Among the latter theorems there is a matrix analogue, applicable to our system when m is zero, of Hille's [5] necessary and sufficient integral equation for a single differential equation to be non-oscillatory for large x while in §5 generalizations of several "integral type" and "limit type" tests of Hille [5] and Wintner [13], [14] for non-oscillation are obtained.

Vector and matrix notation is used throughout; in particular, aside from a few obvious exceptions, we use capital italic letters for $n \times n$ square matrices, German or Greek letters for matrices which are not $n \times n$, and lower case letters for both scalars and vectors. We use the asterisk to denote the transpose of a vector or matrix and the prime to denote the derivative. By the statement that a matrix limit, $\lim_{x \to x_0} A(x)$, exists we mean that the limit of each of its elements exists and is finite; for brevity we write

$$\lim_{b \to \infty} \int_a^b A(x)\, dx = \int_a^\infty A(x)\, dx.$$

The symbols C, C', C'', and D' are employed in connection with vectors and matrices with the usual meanings of continuity and continuity, or piecewise

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