ONE-DIMENSIONAL COHOMOLOGY GROUP OF LOCALLY COMPACT METRICALLY HOMOGENEOUS SPACES

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1. **Introduction.** Let X be a metric space. It is called *metrically homogeneous* if it admits a transitive group of isometries. If such a space X is compact, we can see that it is a projective limit of coset spaces of compact Lie groups which have been thoroughly discussed. However, if X is locally compact, but non-compact, our knowledge is quite limited. Under the further assumption that the group G of all isometries of X is finite-dimensional, Montgomery [8] has established some interesting results on the local properties of X. It is the aim of the present paper to give some information about the global structure of such a space X without restriction on the dimension of G.

The main results are contained in the following two theorems.

Theorem 1. *Let X be a connected, locally compact, separable, metrically homogeneous space (in particular, X is a connected, locally compact group satisfying the first countability axiom). If X is locally connected and non-compact, then X is contractible to a point in the one-point compactification $X' = X + I$ of X.*

Theorem 2. *Suppose that X satisfies all the assumptions of Theorem 1. Then the one-dimensional Lefschetz cohomology group $L^1(X)$ (that is, the usual Čech cohomology group $H^1(X')$ of the one-point compactification X' of X [2]) is either zero or isomorphic with the coefficient group. In the latter case, X is homeomorphic with the product of a real line and a compact space. $L^1(x)$ can also be regarded as the one-dimensional cohomology group with compact supports.*

To illustrate these theorems, we give here two typical examples.

Example 1. Let M be an orientable manifold and x a point of M. $M - x$ has a homogeneous metric only if M is a homotopy sphere, because of Theorem 1.

Example 2. Let P be a Peano continuum with first Betti number greater than one. Then, for any non-cut point x of P, $P - x$ has no homogeneous metric.

2. **Metrically homogeneous spaces.** Let X be a connected, locally compact, separable, metrically homogeneous space. We consider the group G of all the isometries of X. The group G can be topologized in such a manner that G becomes a locally compact, separable metric group [3]. Moreover, G is an effective and transitive transformation group of X in the sense of Montgomery and Zippin. For $x_0 \in X$, let H be the totality of those isometries of X leaving x_0 fixed. Then H is compact [3]. Since G is separable and X of second category,

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