A MODIFIED RIEMANN-STIELTJES INTEGRAL

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The purpose of this note is to introduce a modified notion of RS (Reimann-Stieltjes) integral possessing a combination of desirable properties not all possessed by the usual RS integral or by a generalized RS integral introduced by Pollard [8] and studied by Hildebrandt [5], Graves [4], and others. Specifically, these properties of the integral (of a bounded function f over a closed interval B of q-dimensional Euclidean space relative to an integrator function g) are the following.

(1) The existence of the integral has, when g is positively monotone, a characterization in terms of Darboux upper and lower integrals.

(2) The existence of the integral has, when g is positively monotone, a characterization in terms of zero content (relative to g) of oscillation sets of f.

(3) The existence of the integral has, when g is positively monotone, a characterization in terms of zero measure (relative to g) of the set of discontinuities of f.

(4) If f is integrable relative to g over each of a set of non-overlapping closed intervals $B_1, \cdots, B_n$ whose union is B, then f is integrable relative to g over B and its integral over B is the sum of its integrals over the B_i.

Of these properties the usual RS integral possesses (2) and (3) only, while the Pollard integral possesses (1) and (4) only. Carmichael has established a theorem [2; Theorem III] which seems to imply (1) for the ordinary RS integral. But his upper and lower integrals are not bounds of upper and lower estimates; they are in fact limits of such estimates and may fail to exist. (Compare [4; 270]). It may be remarked that from the proof of our characterization theorem there can be extracted a proof of (2) and (3) for our modified integral which is considerably simpler than Bliss' proof [1] of the corresponding properties for the usual RS integral in one dimension.

1. **Preliminary definitions.** Let R^q denote Euclidean q-space. Let g be a function (throughout, all functions mentioned will be real-valued) on a set D in R^q containing the vertices of an interval I whose closure is

$$\bar{I} = \bigtimes_{i=1}^{q} \{x^i : \alpha^i \leq x^i \leq \beta^i\}.$$

We emphasize that when the word "interval" is used without a qualifying adjective, no restriction is implied; the interval may contain some, all, or none of its faces. For each of the points $v_j = (v_j^1, \cdots, v_j^q)$ $(j = 1, \cdots, 2^q)$ in which each v_j^i is either α^i or β^i, write $n(v_j)$ for the number of α^i's appearing as coordinates

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