SOME CHARACTERIZATIONS OF NORMAL AND PERFECTLY NORMAL SPACES

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We present here first a theorem (Theorem 1) on lattices; then with the aid of this theorem, we prove the characterization theorems (Theorems 2 and 3) for normal and perfectly normal spaces (a normal space is said to be perfectly normal if every closed set in it is the intersection of denumerably many open sets; this definition was given by Čech [2; 40]). Special cases of Theorem 1 for sets and for functions are due to W. Sierpiński, H. Hahn and F. Hausdorff (compare [4] and [3]). For metric spaces, Theorem 2, property (2) is due to Hahn [3; 292], and (3) was first proved by H. Tietze [5]. For Euclidean n-space, Theorem 3, (2) is a classical result of R. Baire [1; 124]; it was generalized to metric spaces by Tietze [5]; (3) of Theorem 3 (for metric spaces) is a theorem of Tietze [5].

Let L be a lattice which can be embedded in a lattice M such that: (1) the l.u.b. and g.l.b. of every denumerable set in L exist in M, and (2) $s \in M$ and $t_n \in L$ imply g.l.b. $\{s \cup t_n\}$ and l.u.b. $\{s \cap t_n\} \in M$ with $s \cup$ g.l.b. $\{t_n\}$ = g.l.b. $\{s \cup t_n\}$ and $s \cap$ l.u.b. $\{t_n\}$ = l.u.b. $\{s \cap t_n\}$. Let L_σ be the set of l.u.b.'s of denumerably many elements in L (L_δ be the set of g.l.b.'s). We now prove

Theorem 1. *If $s \in L_\delta$, $t \in L_\sigma$, and $s \subseteq t$, then there is a $u \in L_\sigma L_\delta$, and $s \subseteq u \subseteq t$.*

Proof. Let s = g.l.b. $\{s_n\}$, and t = l.u.b. $\{t_n\}$. Clearly we may assume $s_n \supseteq s_{n+1}$ and $t_n \subseteq t_{n+1}$ for every n. Let $u_n = (s_1 \cap t_1) \cup (s_2 \cap t_2) \cup \cdots \cup (s_n \cap t_n)$, and $v_n = u_{n-1} \cup s_n$. Let l.u.b. $\{u_n\} = u$ and g.l.b. $\{v_n\} = v$. Now it is readily seen that $v_n \subseteq u \cup s_n$ for every n; hence, g.l.b. $\{v_n\} \subseteq$ g.l.b. $\{u \cup s_n\}$ or $v \subseteq u \cup$ g.l.b. $\{s_n\} = u \cup s$. Since $v_n \supseteq s_n$, therefore, g.l.b. $\{v_n\} \supseteq$ g.l.b. $\{s_n\}$ or $v \supseteq s$. Consequently $v \cup s = v \subseteq u \cup s$. The last statement implies that $v = u \cup s$ since $v \supseteq u$ (this can be seen as follows: $v_n \supseteq u_{n+p}$ for every n and p, hence $v_n \supseteq$ l.u.b. $\{u_{n+p}\}$; finally we have g.l.b. $\{v_n\} \supseteq u$, or $v \supseteq u$). We next show that $u \supseteq s$. For we have u = l.u.b. $\{s_n \cap t_n\} \supseteq$ l.u.b. $\{s \cap t_n\} = s \cap$ l.u.b. $\{t_n\}$ = $s \cap t = s$, since $s \subseteq t$ by hypothesis. Thus $v = u \cup s = u$. So $u \in L_\sigma L_\delta$, and clearly $s \subseteq u \subseteq t$.

In the sequel, unless stated to the contrary, every function is assumed to be real-valued and defined over a topological space R, and $f \leq g$ means $f(x) \leq g(x)$ for every x in R.

Theorem 2. *The following properties about a topological space R are equivalent:*
(1) *R is normal.*

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