THE REPRESENTATION OF FUNCTIONALS BY INTEGRALS

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Let X be a topological space and $C(X)$ be the vector space of all bounded, real-valued, continuous functions over X. The question of whether, given a particular non-negative functional J on $C(X)$, there exists a measure m with respect to which every element of $C(X)$ is measurable for which

$$J(f) = \int f(x)m(dx) \qquad (f \in C(X)), \tag{1}$$

as well as the question of when all non-negative functionals on $C(X)$ satisfy (1), is not answered by the principal theorems concerning such representations. (See Riesz [6], Saks [7], Markoff [5], Kakutani [4], Stone [8].) By means of the construction used by Kakutani, it will be shown that a particular J satisfies (1) if and only if Lebesgue's theorem of monotone convergence, restricted to elements of $C(X)$, holds for J, and that the necessary and sufficient condition that all non-negative J have the form (1) is a particular compactness condition which is itself equivalent to the following conditions:

(a) every increasing sequence of elements of $C(X)$ converging pointwise to an element of $C(X)$ converges uniformly (Dini's theorem);

(b) every continuous function on X is bounded;

(c) every element of $C(X)$ assumes its bound;

(d) every equi-continuous bounded family in the Banach space $C(X)$ has compact closure.

Notation. For any subset E of X, E^- will denote its closure and E' its complement. The function over X assuming the constant value c will also be denoted by c. The characteristic function of a set E will be written k_E, with $k_\Lambda = 0$ (Λ, the empty set). By B^+ we shall mean the set $\{f \mid 0 \leq f \leq 1, f \in C(X)\}$, by E_f the set $\{x \mid f(x) \neq 0\}$.

1. Theorem 1. *The necessary and sufficient condition that* (1) *hold for a non-negative functional J on $C(X)$ is that for $f, f_n \in C(X)$, $n = 1, 2, \cdots, f_n \leq f_{n+1}$ and $f_n(x) \to f(x)$ for all x imply $J(f_n) \to J(f)$.*

Proof. The condition is clearly necessary by Lebesgue's theorem of monotone convergence. Let

$$\mathcal{K} = \{W \mid W = E_f, f \in B^+\}.$$

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