THE BOUNDARY VALUES OF A CLASS OF MEROMORPHIC FUNCTIONS

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1. Introduction. R. Nevanlinna [7] was the first to point out the interest which lies in the class of functions $f(z)$, analytic and bounded in $|z| < 1$, $|f(z)| < 1$, whose radial limit values $\lim_{r \to 1} f(re^{i\theta}) = f^*(e^{i\theta})$ have modulus 1 for almost all θ in $-\pi \leq \theta \leq \pi$. (In the sequel, a function with these properties will be called of *class* (A) in $|z| < 1$.) Following Nevanlinna's work, W. Seidel [10], and G. Hössjer and O. Frostman [4] have made a rather extensive study of functions of class (A) in $|z| < 1$. It was proved independently by Seidel and by Hössjer and Frostman, for example, that if a non-constant function of class (A) omits the value 0 in $|z| < 1$, then there exists at least one radius $\theta = \theta_0$ such that $\lim_{r \to 1} f(re^{i\theta_0}) = 0$. We are concerned in this paper, among other things, with extending this result to functions which are no longer bounded in $|z| < 1$. More precisely, we consider the class of functions which are meromorphic with a finite number of zeros and poles in $|z| < 1$ and whose modulus $|f(re^{i\theta})|$ tends to 1 as $r \to 1$ for almost all θ in $-\pi \leq \theta \leq \pi$. There is an important subclass of such functions which we shall consider whose radial limit values exist almost everywhere; it has been shown by Nevanlinna (see, for example, [6]) that if $f(z)$ is meromorphic with bounded characteristic in $|z| < 1$, then the radial limit values $\lim_{r \to 1} f(re^{i\theta}) = f^*(e^{i\theta})$ exist for almost all θ in $-\pi \leq \theta \leq \pi$, so that for this subclass $|f^*(e^{i\theta})| = 1$ almost everywhere. We show (Theorem 2) that if $f(z)$ is meromorphic of bounded characteristic with at most a finite number of zeros and poles in $|z| < 1$ and if $|f^*(e^{i\theta})| = 1$ almost everywhere, then, unless $f(z)$ reduces identically to a rational function in $|z| < 1$, there exists at least one radius $\theta = \theta_0$ such that $f^*(e^{i\theta_0}) = 0$ or $f^*(e^{i\theta_0}) = \infty$. In the more general case that $f(z)$ is not of bounded characteristic, we show (Theorem 5) that unless $f(z)$ is a rational function, there exists a Jordan arc $\mathfrak{L}$ terminating at a point $e^{i\theta_0}$ of $|z| = 1$ such that as $z \to e^{i\theta_0}$ along $\mathfrak{L}$, $f(z) \to 0$ or $f(z) \to \infty$. In proving Theorem 2, we obtain an interesting result for harmonic functions of bounded mean modulus.

The following definition will be of use in the sequel.

DEFINITION. A function $f(z)$ which is analytic in $|z| < 1$ and whose modulus $|f(re^{i\theta})|$ has radial limit 1 as $r \to 1$ for almost all θ in $-\pi \leq \theta \leq \pi$ will be called of *class* (U) in $|z| < 1$. A function $f(z)$ of class (U) which is of bounded characteristic in $|z| < 1$ will be called of *class* (B) in $|z| < 1$.

2. Harmonic functions and functions of class (B). It follows from the Nevanlinna theory of functions of bounded characteristic in $|z| < 1$ [6; 190] that a necessary and sufficient condition that a function $f(z)$ be of class (B) in

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