SOME INEQUALITIES OF DETERMINANTS OF MINKOWSKI TYPE

By Y. K. Wong

1. Introduction. We consider the determinant of a matrix of the form $I - a$, where a in §4 has non-negative elements and its modulus (defined in §3) is less than unity, whereas in §5, the matrix a has non-negative elements and the sum of all the elements in each column is at most 1. The determinant of the latter type (due to Minkowski, 1900) is a special case of that considered by Ostrowski and Price [4], [7]. Our method differs from the one used by Price and our results are better than his. For the most special cases of our results under the second assumption, we have

$$[1 - a(n, n) - \sum_{j=1}^{n-1} s^{(n)}(j)a(j, n)]D_{n-1} \leq D_n$$

$$\leq [1 - a(n, n) - \sum_{i=1}^{n-1} a(n, i)a(i, n)]D_{n-1} ,$$

where D_n denotes the determinant of $I - a$ (a matrix of order n) and D_{n-1} the determinant of the principal minor consisting of the first $n - 1$ rows and columns of $I - a$. Price's result [7] replaces $s^{(n)}(j) = \sum_{i=1}^{n} a(i, j)$ by 1, and his upper bound is even greater than 1.

Our method involves the notions of moduli and quasi-inverses of matrices. The concept of the moduli of matrices (a term due to E. H. Moore) was introduced by Hilbert in his study of quadratic forms of infinitely many variables. In order to make the paper accessible to a wider class of readers, we give a summary of the properties of the moduli of matrices in §3 with some indication of proofs of certain properties needed in this paper. The notion of quasi-inverse originated in Hilbert's resolvent equation (1909), and Sam Perlis [6] in 1942 formally introduced this concept into the study of radicals in an algebra without unit element. (In order to keep the treatment strictly elementary we do not employ the term quasi-inverse in this paper.) Let a be an element of an algebra; then a_* is a quasi-inverse of a in case $a_* - a = aa_* = a_*a$. One can verify that $1 + a_*$ is the inverse of $1 - a$ if there is a unit element 1, and if a has a quasi-inverse a_*. Returning to the study of finite square matrices, let us consider the bilinear form $x'By$, where B is the inverse of $A = I - a$. If the matrix a is properly restricted, then B may be expressed in the form $I + a_*$, where a_* is the quasi-inverse of a. Thus we have $x'By = x'(I + a_*)y = x'y + x'a_*y = x'y + x'(I + a_*)ay$. Our method centers on the estimation of the row elements of $x'(I + a_*)$. The inequalities are given in (15), (19), (30), (30a) and (30b).

Received August 13, 1951. Presented to the Society, April 28, 1951; delivered as part of the lecture at the Symposium on Linear Inequalities and Programming, sponsored by the Department of Air Force and National Bureau of Standards, Washington, D. C., June 15, 1951. The paper was prepared under Navy contract N6onr-27009.