REPRESENTATIONS OF SEPARABLE ALGEBRAS

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1. **Introduction.** This paper consists of two parts. In §2 we give a new proof of an interesting theorem due to Johnson and Kiokemeister. In §3 we prove the Hilbert space analogue, which appears to be new.

2. **The algebraic theorem.** We need three lemmas which are probably well known. We give the proofs for completeness.

LEMMA 1. *Let A be any ring, S its socle ($=$ union of the minimal left ideals), and M a left A-module satisfying $SM = M$. Then M is a direct sum of irreducible modules isomorphic to minimal left ideals of A.*

Proof. Consider a minimal left ideal L in A, an element x in M, and suppose $Lx \neq 0$. The mapping $c \rightarrow cx$ is an A-homomorphism of L onto Lx. The kernel is a left ideal contained in L and so must be 0. Thus Lx is isomorphic to L. Now $M = SM$ is the union of submodules of the form Lx. This union is cut down to a direct sum by an application of Zorn's lemma.

LEMMA 2. *Let B be the Boolean algebra of all subsets of a countable set, I the ideal of all finite subsets. Then B/I contains a set of pairwise disjoint non-zero elements having the power of continuum.*

Proof. Index the elements of the countable set by the rational numbers. For each irrational number r pick a sequence $x_r = (x_{r1}, x_{r2}, \cdots)$ of rational numbers with $x_{ri} \rightarrow r$. For $r \neq s$, x_r and x_s have only a finite number of elements in common. The elements x_r (or rather their images in B/I) thus constitute the desired set.

Before stating the final lemma, we recall the standard concept of a weak topology. Let A be a ring and M a faithful A-module. The weak topology induced on A by M is defined by taking as a typical neighborhood of 0 the annihilator of a finite subset of M.

LEMMA 3. *Let A be a ring, M a faithful A-module, and N the direct sum of (any number of) copies of M. Then N and M induce the same weak topology on A.*

Proof. It is clear that a neighborhood of 0 induced by M is also one induced by N. Conversely, let $U \subset A$ be the annihilator of $y_1, \cdots, y_r \in N$. Each y_i has a finite number of non-zero components $y_{ij} \in M$ in the representation of N as a direct sum. Then U is also the annihilator of $\{y_{ij}\}$.

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