PARTIAL-SUM COUPLINGS FOR DOUBLE FOURIER SERIES

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1. **Introduction.** Throughout this paper the function $f(t, u)$ is assumed to be Lebesgue integrable over the square $Q(-\pi, \pi; -\pi, \pi)$ and to have period 2π in each variable. The double Fourier series is denoted by $\sigma(f)$ and the rectangular partial sums of $\sigma(f)$ at the point (x, y) are denoted by $s_{mn}(x, y)$. To say that a method of summability S possesses the localization property means that if an integrable function f vanishes in a neighborhood of (x, y), then S sums $\sigma(f)$ at (x, y) to 0. It is well known that ordinary convergence and also the Cesàro method $(C, 1, 1)$ do not possess the localization property. One way to get localization results is to consider restricted limits. If, for any $\lambda \geq 1$, a sequence s_{mn} tends to a limit s when $m, n \to \infty$ in such a manner that $m/n \leq \lambda$, $n/m \leq \lambda$, this limit s being independent of λ, then we say $s_{mn} \to s$ restrictedly; whenever convenient we may denote this by writing $s_{mn} \xrightarrow{r} s$. In a previous paper [1] the author showed that restricted summability (C, α, β) does possess the localization property if and only if $\alpha \geq 1$, $\beta \geq 1$. Thus restricted convergence of $\sigma(f)$ does not possess the localization property. In particular this means that even if (x, y) is a point of continuity of f it may not be true that $s_{mn}(x, y) \to f(x, y)$ restrictedly. It was shown in [1], however, that at a point (x, y) of continuity of f the double Fourier series $\sigma(f)$ is restrictedly summable $(C, 1, 1)$ to $f(x, y)$.

As an alternative to Cesàro means, Rogosinski [2] introduced and studied couplings of the partial sums of a simple Fourier series. In the present paper we shall study the analogous couplings of the partial sums of double Fourier series. For integers p and q and real h and k we define the couplings by means of the equation

$$\begin{aligned} \kappa_{mn}(x, y; p, q; h, k) = \frac{1}{4}\Big[& s_{mn}(x + h, y + k) + (-1)^{p-1} \\ & \cdot s_{mn}\left(x + h + \frac{p\pi}{m}, y + k\right) + (-1)^{q-1} s_{mn}\left(x + h, y + k + \frac{q\pi}{n}\right) \\ & + (-1)^{p+q} s_{mn}\left(x + h + \frac{p\pi}{m}, y + k + \frac{q\pi}{n}\right)\Big]. \end{aligned} \tag{1.01}$$

We note that if p and q are both odd this is a "sum-coupling" and if p or q or both are even we get various "difference-couplings." If $h = h_m = -p\pi/(2m)$ and $k = k_n = -q\pi/(2n)$ we obtain a symmetric coupling which we denote by

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