# NEW CASES OF IRREDUCIBILITY FOR LEGENDRE POLYNOMIALS

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1. **Introduction.** It is well known [2; Chapter 9], [10; Chapter 2] that the Legendre polynomial of degree $n$ can be written in the form

$$P_n(x) = \frac{1}{2^n} \sum_{v=0}^{[\frac{1}{2}n]} (-1)^v \binom{n}{v} \binom{2n-2v}{n-2v} x^{n-2v}. \tag{1}$$

For odd $n$ there is a factor $x$ and the polynomial $L_n(x)$ is introduced as follows

$$L_n(x) = \begin{cases} P_n(x) & (n \text{ even}) \\ x^{-1}P_n(x) & (n \text{ odd}) \end{cases} \tag{2}$$

$$L_n(x) = \frac{1}{2^n} \sum_{v=0}^{m=[\frac{1}{2}n]} (-1)^v \binom{n}{v} \binom{2n-2v}{n-2v} x^{2m-2v}. \tag{3}$$

Although it has been conjectured for many years that $L_n(x)$ for arbitrary $n$ is irreducible in the field of rational numbers, this conjecture remains unproved.

In 1912, J. B. Holt [7] published his first paper concerning this problem. In this paper Holt proves $L_n(x)$ irreducible whenever $n$ lies in the following ranges (in this paper, $p$ denotes an odd prime).

$$2^a \leq n \leq 2^a + 1, \qquad p - 2 \leq n \leq p + 1, \qquad 2p - 2 \leq n \leq 2p - 1. \tag{4}$$

He further demonstrated that $L_n(x)$ has in any case an irreducible factor of degree greater than two-thirds of $n$. In his second paper [8], Holt attempted to extend the ranges of $n$ for which $L_n(x)$ is irreducible to

$$p - 4 \leq n \leq p + 3, \qquad 2p - 4 \leq n \leq 2p - 1. \tag{5}$$

He was successful except for $p + 2$, $p - 3$, and $2p - 3$, in which cases he needed only to exclude the factors $ax^2 + b$. He proved all of these inadmissible for arbitrary $n$ except, oddly enough, a constant times $P_2(x)$. It was left for Hildegard Ille in 1924 to prove in her dissertation [9] that $L_n(x)$ is not divisible by $P_2(x)$. In addition she establishes lower bounds for the degree of any irreducible factor in certain special cases, the irreducibility of $P_n(x)$ if $n = (p-1)p^k$, and the impossibility that the factor of largest degree be another Legendre polynomial. She states without proof that $L_n(x)$ is irreducible for $n$ equal to any of $(p-1)p^k + 1$, $(p-1)p^k + 2$, $(p-1)p^k + 3$. Furthermore, she mentions without proof the following result of Schur, which attests that the Legendre polynomials are quite reducible modulo $p$.

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