THE CHARACTERISTIC ROOTS OF A MATRIX

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In 1900 Bendixson [2] showed that any root $\lambda = x + iy$ of the characteristic equation

$$(1) \qquad | (a_{ij}) - \lambda(\delta_{ij}) | = 0$$

of an $n \times n$ matrix (a_{ij}), when the numbers a_{ij} as well as x and y are real, lies in the rectangle

$$(2) \qquad \alpha_1 \geq x \geq \alpha_n , \qquad | y | \leq [\tfrac{1}{2}n(n-1)]^{\frac{1}{2}} \max \tfrac{1}{2} | a_{ij} - a_{ji} |,$$

where α_1 and α_n are the greatest and least of the characteristic roots of the Hermitian matrix $\frac{1}{2}(a_{ij} + a_{ji})$. Hirsch [8] observed that if in Bendixson's theorem $a_{ij} \pm a_{ji}$ is replaced by $a_{ij} \pm \bar{a}_{ji}$, the first part of (2) holds for arbitrary complex matrix (a_{ij}) and the second part of (2) holds when $(a_{ij} + a_{ji})$ is real. He proved also that

$$(3) \qquad | \lambda | \leq n \max | a_{ij} |.$$

Bromwich [3] extended (2) to the symmetric form

$$(4) \qquad \alpha_1 \geq \frac{1}{2}(\lambda + \bar{\lambda}) \geq \alpha_n , \qquad \beta_1 \geq \frac{1}{2i}(\lambda - \bar{\lambda}) \geq \beta_n ,$$

where α_1, α_n and β_1, β_n are the greatest and least of the characteristic roots of $\frac{1}{2}(a_{ij} + \bar{a}_{ji})$ and $\frac{1}{2}i(a_{ij} - \bar{a}_{ji})$ respectively. I. Schur [10] extended (3) to the symmetric and explicit form

$$(5) \qquad \sum_i | \lambda_i |^2 \leq \sum_{i,j} | a_{ij} |^2,$$

the λ_i's being the characteristic roots of (a_{ij}). But in proving that (4) involves (2), Bromwich had already established (5) for real skew-symmetric (a_{ij}), and it is actually along the same line that Schur derived from (5) and (4) the second part of (2) for real (a_{ij}).

Schur established (5) by writing (a_{ij}) in the form

$$(6) \qquad (a_{ij}) = (u_{ij})(t_{ij})(\bar{u}_{ji}), \qquad ((\bar{u}_{ji}) = (u_{ij})^{-1},\ t_{ii} = \lambda_i ,\ t_{ij} = 0, j > i),$$

and equating the coefficients of λ^{n-1} in the identities

$$| (a_{ij})(\bar{a}_{ji}) - \lambda(\delta_{ij}) | = | (\bar{u}_{ji}) | \, | (a_{ij})(\bar{a}_{ji}) - \lambda(\delta_{ij}) | \, | (u_{ij}) |$$
$$= | (t_{ij})(\bar{t}_{ji}) - \lambda(\delta_{ij}) |,$$

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