ABEL TRANSFORMS OF TAUBERIAN SERIES AND ANALYTIC APPROXIMATION TO CURVES AND FUNCTIONS

BY RALPH PALMER AGNEW

1. **Introduction.** Let $z(t)$ $(= x(t) + iy(t))$ be a continuous complex valued function of the real variable t, $-\infty < t < \infty$, having period L and having bounded variation over each period. Then, as t starts with any given value and increases over an interval of length L, the point $z(t)$ traverses once in the positive direction an oriented closed rectifiable curve C in the complex plane. We suppose that t represents arc length on C and hence that C has length L. This assumption, which we can express in the form

$$\int_0^x | dz(t) | = x \qquad (0 \leq x \leq L),$$

must be kept constantly in mind; as t increases over an interval $a \leq t \leq b$, the point $z(t)$ moves along C, in the positive direction, the distance $(b - a)$.

Let $\sum u_n$ be a series of complex terms satisfying the strong Tauberian condition

$$\lim_{n \to \infty} | nu_n | = h > 0 \tag{1.1}$$

and having partial sums $s_n = u_0 + \cdots + u_n$ all lying on the curve C and progressing steadily along C in the positive direction as n increases. Let $\sigma(r)$ denote the Abel power series transform of $\sum u_n$ so that

$$\sigma(r) = (1 - r) \sum_{k=0}^{\infty} r^k s_k \qquad (0 < r < 1). \tag{1.2}$$

It is our object to study the set of limit points of $\sigma(r)$, by which we mean the set of points ζ representable in the form $\lim \sigma(r_n) = \zeta$ where r_n is a sequence for which $0 < r_n < 1$ and $\lim r_n = 1$. It turns out that this set of limit points is a curve C_h which is uniquely determined by C and the constant h in (1.1), being completely independent of the series used in its definition. We shall obtain equations and some of the properties of these curves C_h. In particular, we find that C_h is always an analytic curve and that, when h is near zero, C_h is an extraordinary analytic approximation to C.

2. **Partial sums.** Since (1.1) implies that $| s_n - s_{n-1} | = | u_n | = o(1)$ as $n \to \infty$ and that $\sum | s_n - s_{n-1} | = \infty$, it follows from our hypotheses that the points s_n traverse the curve C over and over again as n increases. When m and n are positive integers, let $d(m, n)$ denote the distance, which is to be

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