ZEROS OF SIMPLE SETS OF POLYNOMIALS

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1. A sequence $\{p_n(z)\}$ of polynomials is said to be *basic* in the sense of J. M. Whittaker [10; Ch. I, III] if any polynomial can be written as a finite linear combination of the polynomials $\{p_n(z)\}$. Thus if $p_n(z)$ is of degree n the set is necessarily basic and it is called a *simple set.* The mode of increase of a basic set is determined by its order and type as defined by Whittaker [10; 11]. This note deals with the mode of increase of simple sets of polynomials whose zeros lie in a finite circle. Since a substitution $z = kz' + a$ will transform any circle $| z - a | = k$ to the unit circle, we shall assume the zeros of the polynomials to lie within or on the unit circle and apply Whittaker's theorem for linear substitution [10; 12]. It has been shown [6] that if the zeros of the polynomials belonging to simple sets all lie within or on the unit circle the set will be of increase not exceeding order 1 type $\frac{3}{2}$. The number $\frac{3}{2}$ is certainly not the best possible and the exact value ρ was shown by M. T. Eweida [1] to lie between 1.240 and 1.443. My aim in this note is to reduce this range, and, by applying the results of N. Levinson and S. S. Macintyre concerning the zeros of derivatives of integral functions, I was able to show that

$$1.351 < \rho < 1.378. \tag{1.1}$$

Thus the above range is reduced by about 86 per cent. In fact I have proved the following theorem

Theorem 1. *When the zeros of polynomials belonging to a simple set all lie within or on the unit circle the set will be of increase not exceeding order* 1 *type* 1.378.

Thus the upper bound of the range (1.1) is obtained; the lower bound is provided by a particular example. The procedure of the proof also leads to the derivation of the analog of Schoenberg's theorem [8] for the case of real zeros. This is formulated in the following theorem.

Theorem 2. *If the zeros of polynomials belonging to a simple set are real and lie in the segment* $-1 \leq x \leq 1$, *the set will be of increase not exceeding order* 1 *type* $4/\pi$ *and this bound is exact.*

(Note that, since the substitution $z = z'e^{\alpha i}$ does not affect the mode of increase of a basic set [10; 12], this theorem is also true when the zeros lie on any diameter of the unit circle.)

2. *Proof of Theorem* 1. Let $a_1, a_2, \cdots$ be a sequence of points within or on the unit circle and denote, as usual, by $\{G_n(z)\}$ the set of Gontcharoff polynomials (see [2], [3], [10]) associated with the above points. That is to say,

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