LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX. IV: APPLICATIONS TO STOCHASTIC MATRICES

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This paper is a continuation of my papers [1], [2], [3]. The numeration of the theorems and equations will be continued.

A square matrix $A = (a_{\kappa\lambda})$ of order n is called stochastic if all the elements are non-negative and if

$$\sum_{\lambda=1}^{n} a_{\kappa\lambda} = 1 \qquad (\kappa = 1, 2, \cdots, n). \tag{61}$$

If all the elements are positive, then A is called a positive stochastic matrix. The properties of the characteristic roots of stochastic matrices are of importance for the theory of stochastic processes.

It is well known that all the characteristic roots of a stochastic matrix lie in the interior or on the boundary of the unit circle. The point $z = 1$ is always a characteristic root. R. v. Mises [12; 536] pointed out that the results of G. Frobenius [8], [9], [10] on matrices with positive and non-negative elements can be used for stochastic matrices and V. Romanovsky [13] formulated these results for stochastic matrices.

In particular, the following theorem holds. If A is unreduced (see [3; 872]), then $z = 1$ is a simple characteristic root. No other point on the boundary of the unit circle can be a characteristic root unless all the elements of the main diagonal vanish.

Let $a_{\kappa\kappa}$ be the smallest element of the main diagonal. M. Fréchet [6], [7] showed that all the charactersitic roots of a stochastic matrix lie in the interior or on the boundary of the circle

$$| z - a_{\kappa\kappa} | \leq 1 - a_{\kappa\kappa} . \tag{62}$$

Moreover, without using the results of Frobenius, he proved that no point different from 1 on the boundary of the unit circle can be a characteristic root unless at least two of the elements of the main diagonal vanish.

Finally, N. Dmitriev and E. Dynkin [4] proved that no characteristic root of a stochastic matrix of order less than or equal to n can lie in the interior of the segments bounded by the unit circle and the chords joining the point $z = 1$ with the points $z = \exp(2\pi i/n)$ and $z = \exp(-2\pi i/n)$. Any other point of the sector $-2\pi/n \leq \arg z \leq 2\pi/n$ including the chords can be a characteristic root of such a matrix. In a second paper [5] they generalized this result.

In this paper, all the mentioned theorems on characteristic roots of stochastic

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