INDEPENDENCE OF ARITHMETIC FUNCTIONS

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1. Bellman and Shapiro [2] have proved the algebraic independence of the six arithmetic functions m, $\phi(m)$, $d(m)$, $\sigma(m)$, $2^{\nu(m)}$, $\mu(m)$, where $\phi(m)$ denotes the Euler function, $\mu(m)$ the Moebius function, $d(m)$ the number of divisors, $\sigma(m)$ the sum of divisors, $\nu(m)$ the number of distinct primes dividing m. A short, direct proof of the algebraic independence of the first five functions was given by L. I. Wade [3].

The possibility of considering in connection with such questions the Dirichlet rather than the "ordinary" product of arithmetic functions came up in conversation with Wade. By an arithmetic function we shall mean a single valued function whose values are (finite) complex numbers. We recall that the Dirichlet product of two functions is defined by

$$h(m) = \sum_{ab=m} f(a)\, g(b); \tag{1.1}$$

for brevity we write $h = f \cdot g$ or simply fg, when there is no danger of confusion. If we introduce the unit function u defined by

$$u(1) = 1, \qquad u(m) = 0 \qquad (m > 1), \tag{1.2}$$

and define the sum $l = f + g$ by means of $l(m) = f(m) + g(m)$, then it is easily verified that the totality of arithmetic functions constitutes a domain of integrity D; the set of functions cu, where c runs through the complex numbers, forms a field C' isomorphic with the complex field C. (For references, see [1].)

As is familiar there are various relations connecting the functions m, $\phi(m)$, $d(m)$, $\sigma(m)$, $\mu(m)$ as well as certain other well-known arithmetic functions. To put these relations in a convenient form we define the set of functions

$$I_k(m) = m^k \qquad (k = 0, 1, 2, \cdots). \tag{1.3}$$

Then we have

$$\begin{aligned} \mu \cdot I_0 &= u, & d &= I_0 \cdot I_0 = I_0^2\,, \\ \sigma &= I_0 \cdot I_1\,, & \phi &= I_1 \cdot \mu = I_1 \cdot I_0^{-1}, \end{aligned} \tag{1.4}$$

so that the five functions mentioned above are expressed in terms of I_0 , I_1 . We shall now show that there is no algebraic relation connecting I_0 and I_1 . More generally we prove

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