DERIVATION AND COHOMOLOGY IN SIMPLE AND OTHER RINGS. I.

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The theory of derivations in rings has been considered by many authors, including Jacobson [9], Hochschild [4], Malcev [10], either for itself or for application's sake. Its higher-dimensional generalization, that is, the cohomology theory in rings, analogous to the group case of Eilenberg–MacLane [3], has been developed by Hochschild [5], [6], [7]. In order to show that a certain cohomology group is 0, one usually appeals to a theorem which secures the complete reducibility of a certain module. For instance, if A is a central simple algebra, say, over a field Ω, and B is a simple subalgebra of A, then every A-B-double-module, over Ω, is completely reducible. In particular, the A-B-double-module $A \times_\Omega B$ is completely reducible, and this leads to the assertion that every derivation of B into A, over Ω, is inner, in A. The complete reducibility of A-B-double-modules can be seen by considering them as $A' \times B$-right-modules, where A' is inverse-isomorphic to A, and observing that $A' \times B$ is a simple algebra. However, this argument strongly depends on the (finite-)algebra property of A, B, or at least on the finiteness of $(B : \Omega)$, and can not be transferred to the general case where A and B are simple rings, with minimum condition, infinite over their centers. Now, in connection with his recent study [13] of automorphisms in simple rings, the writer showed, though in a little different formulation, that if B is a weakly normal (see §1 below; also see "galoisien" in Dieudonné [2]) simple subring, with minimum condition, of a simple ring with minimum condition A, then the A-B-module A is completely reducible. If C is a second weakly normal simple subring, with minimum condition, of A which is contained in B, it turns out, as we shall see below, that the direct product $A \times_C B$ (in fact $A \times_C A$) is also completely reducible as A-B-module, and this leads to the result that every derivation of B in A vanishing on C is ~ 0 in A (Theorem 2). More generally, if a double-module Q of B has a certain special structure with respect to A, B, C, then every derivation of B in Q vanishing on C is ~ 0 (Theorem 3). However, one has to observe that the assertion depends on that special structure of Q, contrary to the algebra case where the wholesale complete reducibility can be secured. On considering certain modules, introduced by Hochschild, we can obtain similar results also for higher cohomology groups (§5). But, in order to do so, we have first to make a certain preliminary study of those modules (§4), which was unnecessary in case of algebras, again because of the non-wholesale character of our argument.

Although our interest is primarily in rings which are not (finite-)algebras, we want to note that the classical case of simple (and semisimple) algebras "almost"

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