THE LEBESGUE CONSTANTS FOR $(E, 1)$ SUMMATION OF FOURIER SERIES

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1. **Introduction.** It is shown in this article that the n-th Lebesgue constant for the Euler summation method $(E, 1)$, as applied to Fourier series, has the same asymptotic representation (2.3) as that obtained in the case of Borel's exponential means [8], [9].

The divergence of these constants implies, as a consequence of Lebesgue's work on singular integrals [7] (see also [12; 98–99]), the existence of a continuous function whose Fourier series is non-summable $(E, 1)$ for at least one value of the independent variable.

This approach was introduced by Lebesgue [6; 86] in connection with the corresponding phenomenon for convergence.

It is well known that any series summable by any Euler mean (E, k) is also summable in the sense of Borel [5], [3; 183]. Consequently, the existence of a continuous function whose Fourier series is non-summable (E, k) is a corollary of Moore's results [10] first establishing this phenomenon for Borel's method. In fact, Moore pointed this out explicitly for $(E, 1)$.

Prachar and Schmetterer [11] undertook a direct study of the Lebesgue constants for the $(E, 1)$ and $(E, 2)$ methods, proving, in each case, that the n-th Lebesgue constant is exactly of order $\log n$ as n becomes infinite.

The more detailed study of the $(E, 1)$ method presented here is divided in two parts. First, the Borel- and Euler-Lebesgue constants are compared with one another, based on the observation that $\cos t$, which occurs in the Euler case, is an approximation to e^{-t^2}, which occurs in the Borel case. Then the Euler-Lebesgue constants are re-examined by comparison with the Lebesgue constants for convergence. This latter method is the one employed in deriving the asymptotic representation of the Borel-Lebesgue constants [8].

2. **Comparison of the Borel and Euler constants.** The n-th Lebesgue constant for $(E, 1)$ summation of Fourier series can be written [3; 364]

$$L_E(n) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \cos^n t \, \frac{|\sin (n+1)t|}{\sin t} \, dt, \tag{2.1}$$

for Borel summation [12; 186]

$$L_B(x) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \exp(-2x \sin^2 t) \, \frac{|\sin (x \sin 2t + t)|}{\sin t} \, dt. \tag{2.2}$$

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