THE ARITHMETIC OF MÉNAGE NUMBERS

By John Riordan

1. **Introduction.** The problème des ménages, proposed by Lucas in 1891, asks for the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next to his own wife. The ménage numbers, in the first instance, are those of the reduced problem with the positions of wives fixed; more generally, they are the numbers $u_{n,r}$ giving the numbers of arrangements (reduced problem) in which exactly r husbands are next to their own wives.

A collection of the more important results, along with an historical résumé, has been given in [3], but curiously none of these seem apt for a study of the arithmetic of the numbers, that is, of the structure of the residues to a prime modulus. A new inverse relation, developed in §2, is sufficient to disclose this structure, which has simple periodic properties, as is proved in §3.

More precisely, it will be shown that numbers $u_{n,r}$ have period $2p^2$ for every r and that $u_{n+r,r}$ have period p^2 for every n.

2. **Inverse relations.** The simplest expression for the numbers $u_n \equiv u_{n,0}$ seems to be that discovered by Touchard [4], and elegantly proved by Kaplansky [2]; it goes as follows

$$u_n = \sum_0^n (-1)^k \frac{2n}{2n - k} \binom{2n-k}{k} (n-k)!. \tag{1}$$

With $(n - k)! \equiv v_{n-k}$, and the usual conventions of the symbolic calculus ($v^n \equiv v_n$), as noticed by Touchard, this may be written

$$\begin{aligned} u_n &= 2 \cos [2n \cos^{-1} (v^{\frac{1}{2}}/2)] \\ &= 2T_{2n}(v^{\frac{1}{2}}/2) \end{aligned} \tag{1.1}$$

with T_n a Tchebycheff polynomial. Then from the inverse relation for these polynomials [1; 278]; namely,

$$(2 \cos \theta)^{2n} = \sum \binom{2n}{k} 2 \cos (2n - 2k)\theta$$

it follows that

$$n! = \sum_0^n \binom{2n}{k} u_{n-k} \tag{2}$$

with $u_0 = 1$, $u_1 = -1$, $u_2 = 0$ (these numbers have no combinatorial significance).

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