THE $\mathfrak{L}$-CLOSURE OF EIGENFUNCTIONS ASSOCIATED WITH SELF-ADJOINT BOUNDARY VALUE PROBLEMS

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By modifying slightly a procedure used by Kneser [2], [1; Chapter XI] in connection with certain self-adjoint boundary value problems for second order differential equations, a rather simple function-theoretic proof will be given for the closure in the space $\mathfrak{L}$ (all Lebesgue integrable functions) of the eigenfunctions of a self-adjoint, n-th order, boundary value problem on a finite interval. (Earlier references to Cauchy, Poincaré and Stekloff are given in Kneser [2].)

Let D denote the set of all complex-valued functions $x = x(t)$ on a finite interval $a \leq t \leq b$ which have continuous $(n - 1)$-th derivatives and with $x^{(n-1)}$ absolutely continuous. The linear differential operator L is defined for all $x \in D$ by

$$L(x) = p_0 x^{(n)} + p_1 x^{(n-1)} + \cdots + p_n x, \tag{1.0}$$

where the p_i are continuous complex-valued functions of t on $a \leq t \leq b$, and $| p_0(t) | \neq 0$ on $a \leq t \leq b$. Consider n linearly independent boundary conditions

$$U_i(x) = \sum_{j=1}^{n} (a_{ij} x^{(j-1)}(a) + b_{ij} x^{(j-1)}(b)) \qquad (i = 1, \cdots, n), \tag{1.1}$$

where the a_{ij} and b_{ij} are constants. If the relations $U_i(x) = 0$ hold for $i = 1, \cdots, n$, the shorter notation $U(x) = 0$ will be used. The boundary value problem consists in finding those λ for which

$$L(x) = \lambda x, \qquad U(x) = 0 \tag{1.2}$$

have one or more solutions of class C^n on $a \leq t \leq b$. (From $L(x) = \lambda x$ it is immediate that any solution $x \in D$ is of class C^n.) Those values of λ for which (1.2) has solutions (not identically zero) are called eigenvalues and the solutions are called eigenfunctions.

If u, v are measurable functions of t and if the product $u\bar{v}$, where the bar denotes the complex conjugate, is integrable over $a < t < b$ then let

$$(u, v) = \int_a^b u\bar{v}\, dt.$$

Two functions u and v are said to be orthogonal if $(u, v) = 0$. A function u is said to be normalized if $(u, u) = 1$. The problem (1.2) is called self-adjoint if for every $u, v \in D$ for which $U(u) = 0$ and $U(v) = 0$, the relation

$$(L(u), v) = (u, L(v)) \tag{1.3}$$

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