TWO THEOREMS ON SCHLICHT FUNCTIONS

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1. **Introduction.** Let (S) denote the class of functions $w = f(z) = \sum_1^\infty a_n z^n$, $a_1 = 1$, which are regular and schlicht for $|z| < 1$. The most famous problem concerning such functions is whether $|a_n| \leq n$, $n = 2, 3, \cdots$, with equality for any n only for functions having the form $f(z) = z/(1 - ze^{i\theta})^2$. This is the so-called Bieberbach conjecture. It is relatively easy to prove that $|a_2| \leq 2$ and there are many proofs [2] of this result. There are only two essentially different proofs [6], [11] of the inequality $|a_3| \leq 3$, both proofs being based upon recently developed variational methods. No sharp upper bound is known for $n > 3$, although it has been shown [5] that $|a_n| < en$ and that [1] lim sup $(n \to \infty)$ $|a_n|/n < e/2$. There are only two subclasses of (S) for which it has been proven that $|a_n| \leq n$, namely, (a) when all a_n are real [5] and (b) when $f(z)$ maps $|z| < 1$ onto a domain starlike with respect to $w = 0$. The lowest upper bound to date for $|a_4|$ for functions of (S) has been found [4] to be $|a_4| < 4.16$. It is proposed in §2 of this paper to establish

Theorem 1. *Let $f(z) \subset (S)$, then $|a_4| < 4.0891$.*

Because of the laborious, although simple, calculations involved in the proof of Theorem 1, we omit those that are readily verifiable. A method of investigating the conjecture $|a_4| \leq 4$ has been devised by Schaeffer and Spencer [11]; considerable numerical work is involved and computations begun in the winter of 1946–47 are being at present carried out as part of a project sponsored by the office of Naval Research. No results have as yet been announced.

The basic concepts upon which we rely for the proof of Theorem 2 in §3 of this paper will now be stated [10]: Let $K(N^{\frac{1}{2}}i)$ be the complex quadratic extension generated over the rational field by a root of $x^2 + N = 0$, where $N \geq 1$ is a rational integer containing no square factor. If $-N \equiv 1 \pmod 4$ the integers of $K(N^{\frac{1}{2}}i)$ are $\alpha = \frac{1}{2}(s + tN^{\frac{1}{2}}i)$, where s and t are both even or both odd rational integers. If $-N \equiv 2$ or $3 \pmod 4$ the integers are $\alpha = s + tN^{\frac{1}{2}}i$, where s and t are rational integers. The units of $K(N^{\frac{1}{2}}i)$ are those integers whose norm $|\alpha|^2 = 1$. The only units of $K(N^{\frac{1}{2}}i)$ are ± 1 except for $N = 1, 3$ in which case the units are ± 1, $\pm i$ and ± 1, $\pm p$, $\pm p^*$ respectively, where $p = \frac{1}{2}(-1 + 3^{\frac{1}{2}}i)$ and p^* is the complex conjugate of p. The only fields $K(N^{\frac{1}{2}}i)$ in which norm $\alpha = |\alpha|^2 = 2$ are $K(i)$ and $K(2^{\frac{1}{2}}i)$. Similarly norm $\alpha = |\alpha|^2 = 3$ can occur only in $K(3^{\frac{1}{2}}i)$. There exists in every real quadratic realm $K(N^{\frac{1}{2}})$ an infinite number of units η, and a unit $\epsilon > 1$ such that every unit η of the realm has the form $\eta = \pm\epsilon^k$, where k is a positive or a negative rational integer or zero. We denote by $K(1)$ the field of rationals. Let $S(K(N^{\frac{1}{2}}i))$ denote the class

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