L^2-APPROXIMATION BY PARTIAL SUMS OF ORTHOGONAL DEVELOPMENTS

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Let $\{\phi_n\}$, $n = 1, 2, 3, \cdots$, be an orthonormal set of functions defined on the unit interval, that is,

$$\int_0^1 \phi_n(t)\phi_m(t)\, dt = \delta_{mn} .$$

Such a set will be called an ON, and a CON if it is complete in L^2.

Let $s_n = s_n(f; x)$ be the n-th partial sum of the series

$$\sum_{n=1}^{\infty} \phi_n(x) \int_0^1 f(t)\phi_n(t)\, dt,$$

and define

$$\| f \| = \left(\int_0^1 f^2(t)\, dt\right)^{\frac{1}{2}}.$$

Let $V(f)$ be the total variation of f on [0, 1]. We say that $f \varepsilon BV$ if $V(f) < \infty$, and that $f \varepsilon$ Lip 1 if

$$N(f) = \text{l.u.b.} \left| \frac{f(x) - f(t)}{x - t} \right| < \infty \qquad (0 \leq x \leq 1, 0 \leq t \leq 1). \tag{1}$$

Our main results are contained in Theorems 1 and 2, and may be stated as follows. For every CON, the least upper bound of $\| f - s_n \|$, taken over all f such that $V(f) = 1$, tends to zero not faster than $n^{-\frac{1}{2}}$; if we take the least upper bound over all f such that $N(f) = 1$, the corresponding order of magnitude is n^{-1}. In both cases, consideration of the trigonometric system shows that these estimates are the best possible.

The proofs of the above facts easily yield estimates concerning the total variations and the Lipschitz constants of members of ON sets (Theorem 3). Some other consequences are considered in Theorems 4 and 5.

We now introduce the notation

$$\mu_n = \text{l.u.b.} \frac{\| f - s_n \|}{V(f)} \qquad (0 < V(f) < \infty), \tag{2}$$

$$\lambda_n = \text{l.u.b.} \frac{\| f - s_n \|}{N(f)} \qquad (0 < N(f) < \infty). \tag{3}$$

Evidently μ_n and λ_n depend on $\{\phi_n\}$.

By the cosine set we mean the functions $1, 2^{\frac{1}{2}} \cos \pi t, 2^{\frac{1}{2}} \cos 2\pi t, \cdots$.

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