## CONVEX SETS IN LINEAR SPACES. II.

By V. L. Klee, Jr.

**Introduction.** This paper is a sequel to the author's *Convex sets in linear spaces* [4]. Notation and terminology of [4] are used without further explanation.

In §1 it is demonstrated that for a convex subset $X$ of an arbitrary linear system $L$, the basic questions concerning polygonal connectedness of $L \sim X$ have the same answers as in the case of a two-dimensional $L$. §2 contains some characterizations of hyperplanes. §3 supplies a proof, in answer to a question of Erdös, that Hilbert space cannot be covered by fewer than $\mathbf{c}$ hyperplanes. In §4 it is proved that every non-reflexive separable Banach space contains a pair of disjoint bounded closed convex sets which cannot be separated by a hyperplane. This extends a result of Dieudonné [2] and, when combined with Tukey's separation theorem [10] for weakly compact sets, provides a new characterization of reflexivity. In §5 there is described a topologization (due to Erdös) of the real number system which answers affirmatively the question $(Q_1)$ of [4].

In addition to the notation of [4], we here use $X \oplus Y$ to denote the smallest linear subspace containing $X \cup Y$, and $X \oplus \equiv X \oplus X$.

1. **Polygonal connectedness of the complement of a convex set.** The theorems of this section are quite elementary. It is hoped that the accumulation of enough results of this general nature may throw some light on the question $(Q_2)$ [4;447]. Also, two of these results provide characterizations of hyperplanes in §2.

*Standing hypotheses in* §1: *$L$ is a linear system and $X$ is a convex subset of $L$.*

(1.1) *Suppose $X$ is a convex cone and $L \sim X$ is not polygonally connected. Then $X$ is a maximal variety.*

*Proof.* Clearly the complement of every non-maximal variety is polygonally connected, so it suffices here to prove that $X$ is a variety. We suppose without loss of generality that $\phi$ is the vertex of $X$. Then if $X$ is not a variety there is a point $x \in X$ for which $-x \notin X$. Now consider an arbitrary $\lambda \in [0, 1]$ and $y \in L \sim X$. If $\lambda(-x) + (1 - \lambda)y \in X$ then (since $X$ is closed under addition and under multiplication by positive scalars) $y \in X$. Hence, $[-x, y] \subset L \sim X$ for each $y \in L \sim X$ and $L \sim X$ is 2-gonally connected, which is a contradiction. Thus $X$ is a variety and the proof is complete.

(1.2) *Suppose $L \sim X$ is not polygonally connected. Then if $X_0 = \Lambda$, $X$ is a maximal variety. If $X_0 \neq \Lambda$, $X$ is the "strip" between two parallel maximal*

Received July 20, 1951.