AN EXTENSION OF THE SUM THEOREM OF DIMENSION THEORY

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1. **Introduction.** The purpose of this paper is to prove for a class of spaces more general than separable metric the sum theorem of dimension theory, using the Urysohn-Menger dimension function. This class of "K-separable" spaces is defined and some of its properties are developed. It is proved that the property of being K-separable and metric is hereditary, additive, and topological. An example of a nowhere separable metric space by Urysohn is shown to be K-separable and examples of non-K-separable spaces are given. The equivalence of the Urysohn-Menger dimension function $d_1(x)$ and the dimension function $d_2(x)$ (separation of disjoint closed sets) for K-separable spaces is proved together with some other equivalences and theorems. Finally, the Sum and Decomposition Theorems for n-dimensional sets are proved in K-separable spaces.

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2. **Notation.** Lower case Roman letters used as subscripts will denote a countable range; lower case Greek letters used as superscripts will denote an uncountable range. The symbol $\{A \mid P(A)\}$ will denote the set of all A such that the proposition $P(A)$ is true, and $\bigcup \{A \mid P(A)\}$ the union of the sets A such that property $P(A)$ holds.

3. **Definitions.**

(3.1) *Covering.* A covering of a space X is a collection (possibly uncountable) of open sets whose sum is X.

(3.2) *Point basis.* If $\mathfrak{U}$ is a collection of open sets of a space X, then $\mathfrak{U}$ is a point basis of $x \in X$ provided that for every open set O containing x there exists some $U \in \mathfrak{U}$ such that $x \in U \subset O$.

(3.3) *Separable at a point.* A space X is separable at a point p if there exists an open set O containing p such that O is separable.

(3.4) *Nowhere separable.* A space X is nowhere separable if X is not separable at any of its points.

(3.5) *δ-void.* A metric space is δ-void if $x, x' \in X$ implies $\rho(x, x') \geq \delta$. If $X = p$ is a single point then p is 1-void.

(3.6) $\eta(\epsilon)$. For $\epsilon > 0$ define

$$\eta(\epsilon) = \max \{1/k \mid 1/k \leq \epsilon\} \qquad (k = 1, 2, \cdots).$$

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