THE FUNDAMENTAL SOLUTION OF $\Delta\psi + e(y)\psi_y = 0$

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Introduction. The *fundamental solution* of a linear partial differential equation of second order

$$\Delta\psi + d\psi_x + e\psi_y + f\psi = 0,$$

is a solution of the form

$$\psi = P(x, y) \log r + W(x, y),$$

$$P(x_0, y_0) \neq 0, \qquad r = ((x - x_0)^2 + (y - y_0)^2)^{\frac{1}{2}},$$

in which P, W are regular at the point (x_0, y_0), and represents the generalization (for further details, see [5; vol. 3, pp. 230–234]) of the logarithmic potential $\log r$, associated with Laplace's equation, to the general second order partial differential equation of elliptic type. The existence of such a solution was first demonstrated by Hilbert when the coefficients d, e, f are analytic functions of x, y. (See [10; 515, 570], where mention is made of the work of E. R. Hedrick; for recent work on the existence of the fundamental solution for elliptic equations of higher order in more than two independent variables, see [7], where references are given to the work of Hadamard, Somigliana, Fredholm, Hergoltz, Bureau, Kodaira, as well as to the work of Levi, who deals with partial differential equations in two independent variables with coefficients not necessarily analytic functions.) An explicit formula in closed form for the fundamental solutions valid in the small was first given by Tricomi [11; 134] for the case

$$d = 0, \qquad e = \frac{p}{y}, \qquad f = 0 \qquad\qquad (p = \tfrac{1}{3})$$

and only recently by Weinstein [12; 111], [13] for an arbitrary constant p. Weinstein employs the methods of his generalized axially symmetric potential theory and gains a formula for the fundamental solution involving the Legendre Q-function valid as long as $y \neq 0$.

In this paper we propose a different method which leads to an explicit formula in closed form for the fundamental solution of $\Delta\psi + e(y)\psi_y = 0$, not only for the case handled by Weinstein, but also for a more extended class of functions $e(y)$. New independent variables

$$\xi = \xi(r, y), \qquad \eta = y,$$

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