THE REPRESENTATION OF FUNCTIONS OF BOUNDED VARIATION BY SINGULAR INTEGRALS

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1. **Introduction.** In the first part of this note we shall prove a theorem concerning the following equation

$$\lim_{\lambda \to \infty} \int_E f(t)\phi(t - x, \lambda)\, dt = \frac{1}{2}\left\{f(x + 0) + f(x - 0)\right\}, \tag{1}$$

where E is a linear measurable set containing $t = x$ as its point of density, $f(t)$ is of bounded variation *in the wide sense* on E. (See, for example, [5; 221]) and $\phi(t, \lambda)$ is a kernel function satisfying certain general conditions. Though a rather complete discussion on general singular integrals had already been given by Lebesgue in his famous paper [3], a particular investigation of (1) seems still necessary in view of the fact that we are dealing with the representation of functions of bounded variation in the wide sense and our singular integral is defined over a measurable set. However, in establishing a theorem about (1) we must make use of certain results of Lebesgue.

In the second part we shall discuss the following relation

$$\lim_{\lambda \to \infty} \int_E f(t) \frac{\phi(\lambda(t - x))}{t - x}\, dt = f(x + 0) \int_0^\infty \frac{\phi(t)}{t}\, dt - f(x - 0) \int_0^\infty \frac{\phi(-t)}{t}\, dt. \tag{2}$$

This is obviously a specialization of (1), and includes the classical equation for Dirichlet's integral as a particular case with $\phi(\lambda u) = \sin \lambda u$, $E = (a, b)$. It will be seen that such a specialization implies several consequences.

The following definition will be found useful in the next two sections:

Definition. *Let $\chi(u)$ denote the characteristic function of E. Then*

$$\rho(t_0, t) = \frac{1}{t - t_0} \int_{t_0}^{t} \chi(u)\, du \qquad (t \neq t_0) \tag{3}$$

is called the average measure of E in (t_0, t). Moreover, if $\rho(t_0, t) \uparrow 1$ as $t \to t_0$, then t_0 is called the point of increasing density of E. For the case where t_0 is a left (or right) boundary point of E, we assume only $t \to t_0 +$ (or $t \to t_0 -$).

2. **A representation theorem and its proof.** We shall use M to denote a fixed upper bound which does not depend on any parameters contained in the related expression. For convenience, let us define:

Class $\mathfrak{K}$. Any kernel function $\phi(t, \lambda)$ defined on $-l \leq t \leq l$ is said to belong to the class $\mathfrak{K}$ if it satisfies 5 *conditions*:

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