A THEOREM ON CYCLIC MATRICES

By Joseph A. Silva

It is well known [1; 444] that if A is a cyclic matrix of order n; $A = \| a_{j-i+1} \|$; $i, j = 1, 2, \cdots, n$; $a_r = a_s$ for $r \equiv s \pmod{n}$, then its determinant is given by

$$d(A) = \prod_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^{j-1} a_j , \tag{1}$$

where the α_i run through the n-th roots of unity.

The standard proof of (1) breaks down when the a_j belong to a field of characteristic p, $p \mid n$. In attempting to carry over this method to the general case the writer was led to the following Theorem 1. Theorem 2 below, which is an easy consequence of Theorem 1, generalizes (1).

Theorem 1. *Let* $A = \| A_{j-i+1} \|$; $i, j = 1, 2, \cdots, n$; $A_r = A_s$ *for* $r \equiv s$ $\pmod{n}$ *be a cylic matrix of order* n *in the* A_r, $r = 1, 2, \cdots, n$. *The* A_r *are square matrices of order* $n_1 \geq 1$; *the elements of* A *are indeterminates. Let* p *be a rational prime and put* $n = p^t m$, $p \nmid m$. *Then*

$$d(A) = [d(D)]^{p^t} \qquad \pmod{p},$$

where $D = \| D_{j-i+1} \|$; $i, j = 1, 2, \cdots, m$; $D_r = D_s$ *for* $r \equiv s \pmod{m}$, *is a cyclic matrix of order* m *in the* D_r. *The* D_r *are themselves matrices of order* n_1 *given by*

$$D_r \equiv \sum_{s=0}^{p^t-1} A_{sm+r} \qquad (r = 1, 2, \cdots, m).$$

Proof. For $t = 0$ the theorem is obvious. Assume $t > 0$ and put $n = pm_1$. Partition A into p^2 square submatrices each of order m_1 in the A_r. Note that A is cyclic in these submatrices; in fact, we have $A = \| A'_{j-i+1} \|$; $i, j = 1, 2, \cdots, p$, where the A'_r are square matrices of order m_1 in the A_r and $A'_r = A'_s$ for $r \equiv s \pmod{p}$. We now put

$$\| C_{ij} \| = \left\| \binom{j-1}{i-1} I \right\| \cdot \| A'_{j-i+1} \| \cdot \left\| \binom{p-i}{p-j} I \right\| \quad (i, j = 1, 2, \cdots, p), \tag{2}$$

where $\binom{r}{s}$ denotes the binomial coefficient $r(r-1) \cdots (r-s+1)/s!$ and I is the unit matrix of order $n_1 m_1$. The C_{ij} will then be square matrices of order $n_1 m_1$ given by

$$C_{ij} = \sum_{l=1}^{p} \sum_{k=1}^{p} \binom{k-1}{i-1} \binom{p-l}{p-j} A'_{l-k+1} \qquad (i, j = 1, 2, \cdots, p).$$

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